THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 10, Number 1/2009, pp. 000–000

ON THE LAGRANGIAN RHEONOMIC MECHANICAL SYSTEMS

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One defines the notion of Rheonomic Lagrangian Mechanical system $\Sigma = (\mathrm{M}, \mathrm{L}, \mathrm{F_e})$ where $L^n = (M, L)$ is a time -dependent Lagrange space and $\mathrm{F_e}$ are the external forces. The evolution equations of $\sum$ are the Lagrange equations (17). The most important result is given by the following Theorem: there exists a canonical semispray *S*, determined only by the system $\sum$, whose integral curves are evolution curves of $\sum$. *S* is a dynamical system on the phase space *Rx TM*. The geometry of the pair (*Rx TM, S*) is the Geometry of Lagrangian rheonomic mechanical system $\sum$.

Key words: Lagrangian rheonomic mechanical system; semispray; nonlinear connection.

1. INTRODUCTION

The problem of geometrization of classical non-conservative mechanical systems is an old one [1, 6, 14]. Essential contributions to solved this problem have been done by R. Abraham and J. Marsden [1], J. Klein, F. Pirani, M. de Leon, O. Krupkova [11], R. Miron and M. Anastasiei [9,11], V. Vujkovici [14], K. (Stevanovic) Hedrih [6] et alt.

Recently, to the 40th Symposium on Finsler Geometry, organized by H. Shimada and S. Sabau in Sapporo 2005, the first author of the present paper solved the problem for the scleronomic non-conservative mechanical system [11]. With this occasion have been defined the Finslerian and Lagrangian mechanical systems and has been determined the canonical evolution semispray for such systems.

In the present paper we consider rheonomic nonconservative mechanical systems by using the geometrical theory of time-dependent Lagrange spaces realized by M. Anastasiei and H. Kawaguchi, [2]. In the particular case of the rheonomic Finslerian mechanical systems we refer to the paper of C. Frigioiu [5].

Now, we study the rheonomic Lagrangian mechanical systems:

$$\Sigma = \left(M, L(t,x,\dot{x}), F_e(t,x,\dot{x})\right)$$

where *M* is the configuration space, $L(t,x,\dot{x})$ is a time-dependent regular Lagrangian and $F_e(t,x,\dot{x})$ are the external forces.

We find the canonical semispray *S* of $\sum$ where its integral curves are the evolution curves of $\sum$. Therefore: the geometry of the vector field *S* on the phase space *Rx TM* is the geometry of the mechanical system $\sum$. But *S* is a dynamical system on *Rx TM* determined only by system $\sum$. So we can study the global properties of $\sum$ such as the stability of solutions curves of the evolution equations of $\sum$.

2. RHEONOMIC LAGRANGE SPACES. PRELIMINARIES

Let M be a real n-dimensional smooth manifold and (TM, π, M) its tangent bundle. Consider the manifold $E=Rx\ TM$ with local coordinates (t, x^i, y^i), ($i, j, k,\ldots= 1, 2,\ldots,n$). These coordinates transform by rule:

$$\tilde{x}^i = \tilde{x}^i(x^1,\ldots,x^n), \qquad \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j \ , \quad \tilde{t} = \Phi(t) \tag{1}$$

with:

$$rank\ (\frac{\partial \tilde{x}^i}{\partial x^j}) = n \ , \qquad \frac{d\Phi}{dt} \neq 0 \,. \tag{2}$$

Usually, we consider $\Phi(t) = a\,t + b, a \neq 0$. Then a rheonomic Lagrange space [2] is a pair:

$$L^n = (M, L(t,x,y))$$

in which $L(t, x, y)$ is a time-dependent regular Lagrangian.

The fundamental tensor of the space L^n is as follows:

$$g_{ij}(t, x, y) = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j} \ , \tag{3}$$

assuming that g_{ij} has a constant *signature* and it is verified the condition:

$$rank(g_{ij}) = n. \tag{4}$$

Therefore we can consider the contravariant tensor g^{ij}.

Now let us investigate the integral of action of the Lagrangian L along a smooth curve $c : [0,1] \to M$:

$$I(c) = \int_0^1 L(\tau, x(\tau), \frac{dx}{d\tau}) d\tau$$

It leads to the Euler-Lagrange equations [2]:

$$\frac{\partial L}{\partial x^i} - \frac{d}{d\tau}(\frac{\partial L}{\partial y^i}) = 0 \,, \qquad y^i = \frac{dx^i}{d\tau} \,. \tag{5}$$

These equations are equivalent to:

$$\frac{d^2 x^i}{d\tau^2} + 2G^i(\tau, x, \frac{dx}{d\tau}) + G_0^i(\tau, x, \frac{dx}{d\tau}) = 0 \tag{6}$$

where:

$$\begin{aligned} 2G^i &= \frac{1}{2} g^{ik} (\frac{\partial^2 L}{\partial y^k \partial x^j} y^j - \frac{\partial L}{\partial x^k}) \ , \quad y^i = \frac{dx^i}{d\tau} \ , \\ G_0^i &= \frac{1}{2} g^{ik} \frac{\partial^2 L}{\partial y^k \partial t} \end{aligned} \tag{7}$$

Clearly, G_0^i is a d-vector field [2]. With respect to (1) the equations (5) and (6) have a geometrical meaning.

Let us consider the vector field $\overset{o}{S}$ on $E = Rx\ TM$:

$$\overset{o}{S} = y^i \frac{\partial}{\partial x^i} - (2G^i + G_0^i)\frac{\partial}{\partial y^i} \tag{8}$$

Then we have [2]:

Theorem 2.1. The following properties hold:

1. $\overset{o}{S}$ *is a semispray on the manifold E.*

2. $\overset{o}{S}$ *depends on the Lagrange space only.*

3. The integral curves of $\overset{o}{S}$ *are given by Euler-Lagrange equations of L.*

Consequently $\overset{o}{S}$ is the dynamical system on the phase space E of the time-dependent Lagrangian L.

Consider the energy of Lagrangian L:

$$E_L = y^i \frac{\partial L}{\partial y^i} - L, \tag{9}$$

and the Poincaré 1-form:

$$\omega_L = \frac{\partial L}{\partial y^i} dx^i - E_L dt \tag{10}$$

Applying the exterior operator d of differentiation we obtain the Cartan 2-form:

$$\theta_L = d\omega_L = \left\{ d\frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} dt \right\} \wedge \{dx^i - y^i dt\} \tag{11}$$

A vector field X on E with the property $i_X \ \theta_L = 0$ is called *characteristic* for the 2-form θ_L.

One proves [2, 9]:

Theorem 2.2. *The semi-spray* $\overset{0}{S}$ *is a characteristic vector field for the Cartan 2-form* θ_L.

Remarking that the system of vector fields $\{ \frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n}, \frac{\partial}{\partial t} \}$ determine the vertical distribution V on E we can consider a splitting of the tangent space $T_u E$:

$$T_u \overset{0}{E} = N_u \oplus V_u \ , \ u \in E. \tag{12}$$

Therefore the horizontal distribution $\overset{0}{N}$ is a nonlinear connection on the manifold E and the adapted basis of the direct decomposition (12) is $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t})$, where:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \overset{0}{N}{}^j{}_i \frac{\partial}{\partial y^j} - \overset{0}{N}{}_i \frac{\partial}{\partial t}. \tag{13}$$

The pair $\left(\overset{0}{N}{}^i{}_j, \overset{0}{N}{}_j \right)$ is the system of coefficients of the nonlinear connection $\overset{0}{N}$.

Theorem 2. 3. Given the semispray $\overset{0}{S}$ of the rheonomic Lagrange space L^n with the coefficients $\left(G^i, G_0^i\right)$ from (7) then there exists a nonlinear connection $\overset{0}{N}$ determined only by L^n. The coefficients of $\overset{0}{N}$ are expressed by:

$$\overset{0}{N}{}^{i}_{j} = \frac{\partial G^i}{\partial y^j} \quad , \quad \overset{0}{N}_i = g_{ij} G^j_0 \ . \tag{14}$$

Remark. In the particular case of time-dependent Finsler space $L = F^2$ the previous theory leads to the geometry of rheonomic Finsler spaces [2].

3. RHEONOMIC LAGRANGIAN MECHANICAL SYSTEMS

A rheonomic Lagrangian mechanical system is a triple:

$$\Sigma = (M, L(t,x,y), F_e(t,x,y)) \ , \tag{15}$$

where $L^n = (M, L(t, x, y))$ is a rheonomic Lagrange space and F_e are the external forces, a priori given as a vertical vector field, in the form:

$$F_e(t,x,y) = F^i(t,x,y)\frac{\partial}{\partial y^i} \tag{16}$$

Consequently, $F^i\,(t, x, y)$ is a d-vector field on the manifold E.

Postulate: The evolution equations of the mechanical system Σ on the phase space $E = TMxR$ are the following *Lagrange equations*:

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial y^i} = F_i, \quad y^i = \frac{dx^i}{dt}, \tag{17}$$

with:

$$F_i(t, x, y) = g_{ij}\, F^j(t, x, y). \tag{18}$$

The previous d-covector field is the d-covariant vector of external forces.

Obviously, if the time t does not explicitly enters in the system Σ we obtain the scleronomic non-conservative mechanical system, studied by the first author in the paper [11].

Two remarkable particular cases are given by:

a. L^n is a Riemannian (or pseudo-Riemannian) space when Σ is the classical nonconservative mechanical systems and (17) are its evolution equations.
b. L^n is a time-dependent Finsler space $F^n = (M, F(t,x,y))$ studied by M. Anastasiei and H. Kawaguchi in [2, I, II, III]. Then Σ is a rheonomic Finslerian mechanical system.

More general, when L is a time-dependent Lagrangian we have the rheonomic Lagrangian mechanical system and when instead of L^n we consider the generalized Lagrange space $GL^n = (M, g_{ij}\ (t, x, y))$ then we have generalized rheonomic mechanical systems, [2,10,11].

Returning to the rheonomic Lagrangian mechanical systems (15) we remark that the Lagrange equations (17) are equivalent to the system of differential equations:

$$\frac{d^2x^i}{dt^2} + 2G^i(t,x,\dot{x}) + G^i{}_0(t,x.\dot{x}) = -\frac{1}{2}F^i(t,x,\dot{x}) \ , \tag{19}$$

where:

$$2G^i = \frac{1}{2}g^{ik}\left(\frac{\partial^2 L}{\partial y^k \partial x^j}y^j - \frac{\partial L}{\partial x^k}\right), \quad G^i_0 = \frac{1}{2}g^{ik}\frac{\partial^2 L}{\partial y^k \partial t} \tag{20}$$

Therefore the geometry of a rheonomic Lagrangian mechanical system Σ is the geometry of the semispray S whose the integral curves are given by the equations (19) and (20).

Example 3.1. The rheonomic Lagrangian mechanical system of time-dependent electrodynamics is given by the Lagrangian:

$$L(t,x,y) = mcg_{ij}(t,x)\dot{x}^i\dot{x}^j + \frac{2e}{mc}A_i(t,x)\dot{x}^i + U(t,x) \tag{21}$$

where *m, c, e* are the known physical constants, $g_{ij}(t, x)$ are the time dependent gravitational potentials, $A_i(t, x)$ are the electromagnetic time dependent potentials and $U(t, x)$ is a time dependent potential function, [2]. The evolution equations (17) can be written without difficulty. In this case the external forces can be given by
$F^i = h(t)y^i$, $h(t)$ being a function depending by *t* only.

Returning to the general theory of the mechanical system Σ, we remark the geometrical meaning of the Lagrange equations (17) or (18) which can be easily demonstrated.

The most important result on the rheonomic Lagrangian mechanical systems Σ is given by the following theorem:

Theorem 3.2. The following properties hold:

1. *There exists a semispray S on the phase space E=RxTM depending only on the rheonomic mechanical system* Σ *from* (15).
2. *S is given by*:

$$S = y^i\frac{\partial}{\partial x^i} - \left(2G^i + G_0^i\right)\frac{\partial}{\partial y^i} + \frac{1}{2}F^i\frac{\partial}{\partial y^i} \tag{22}$$

3. *The integral curves of semispray S are given by the Lagrange equations* (17).

Proof. Writing *S* in the form:

$$S = \overset{o}{S} + \frac{1}{2}F^i\frac{\partial}{\partial y^i}, \tag{23}$$

where $\overset{o}{S}$ is the canonical semispray of rehonomic Lagrange space L^n all properties expressed in the previous Theorem can be proved without difficulties.

From these reasons we can call *S* the canonical semispray of rheonomic mechanical system Σ. It can be developed by the same methods as in the scleronomic case, [11].

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Received October 10, 2008