THE PUBLISHING HOUSE OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 10, Number 1/2009, pp. 000–000

TRADING PRICES WHEN THE INITIAL WEALTH IS RANDOM

Gheorghiță ZBĂGANU [(1)], Marius RĂDULESCU [(2)]

[1] Faculty of Mathematics and Computer Science, University of Bucharest, Academiei 14, Bucharest, RO-010014, ROMANIA, email: zbagang@fmi.unibuc.ro
[2] Institute of Mathematical Statistics and Applied Mathematics, Casa Academiei Române, Calea 13 Septembrie nr.13, Bucharest 5, RO-050711, ROMANIA, email: mradulescu.csmro@yahoo.com
Corresponding author: Gheorghiță ZBĂGANU

The aim of the paper is to study trading prices in the case the initial wealth of the decision maker is random. A closed form for these prices in the case of CARA utilities is given. Order preserving properties of translation type operators acting on the space of power utilities are also investigated.

Key words: utility function, expected utility, selling price, buying price, CARA utilities, decision maker.

1. INTRODUCTION

A *utility function* or simply a *utility* is any continuous increasing function u: $I \to \Re$ where $I \subseteq \Re$ is an open interval. The utility theory asserts that any decision maker is endowed – possibly without being aware of it – with such a function u with the meaning $u(x)$ = "how large is the utility of a gain of x monetary units for the decision maker". This idea has been used especially in microeconomics for a long while. A lot has been written about it. The reader can consult for instance [1], [2], [6], [7] or search on the internet for "utility theory". The Principle of Expected Utility or, Expected Utility Model assumes that a decision maker is always indifferent between a random quantity of money X and its *certainty equivalent* x^* defined by the solution of the equation $\mathrm{E}u(X) = u(x^*)$. One can easily see that

$$x^* = u^{-1}(\mathrm{E}u(X)) \tag{1.1}$$

In spite of the fact that Expected Utility Theory has been much criticized by economists, a lot of research has been done in this framework (see for instance [6] or [11]). The reason is that the model is very nice from a mathematical point of view and has a lot of practical applications. The main concepts of this model are those of ***risky asset*** **(**or ***lottery*****),** ***selling price*** and ***purchasing price*** **(**or ***buying price*****).** We shall also call the selling price and the buying price, trading prices. In an early stage of its development a common assumption in the Utility Theory was that the decision maker possess a deterministic wealth. Recent results and references concerning this approach can be found in [10], [12]-[14]. Further developments of the Utility Theory replaced this assumption with a more realistic one, the initial wealth of the decision maker is a random variable (see [5], [8], [9]).

A ***risky asset*** is any random variable. In the sequel we will assume that we are on some probability space $(\Omega, \mathcal{K}, P)$. In the Expected Utility Model the decision maker has already a deterministic wealth, a, measured in monetary units. He can act as a **seller** or as a **purchaser**.

Suppose that we are in the first situation: the decision maker (now the seller) has a utility function u and a risky asset X and a deterministic wealth a. His expected utility before selling the risky asset is $\mathrm{E}u(a + X)$. He intends to obtain a price H for his risky asset such $\mathrm{E}u(a + X) \le u(a + H)$. The reason of this inequality is obvious: he sells his asset X only if he feels that his certainty equivalent will not decrease after the transaction; otherwise he will prefer to keep his risky asset for himself. The smallest price he will accept would be the solution of the equation

$$\mathrm{E}u(a + X) = u(a + H) \tag{1.2}$$

One can easily see from the above equation that $H = u^{-1}(\mathrm{E}u(a + X)) - a$. The number H is called ***the selling price*** *of X with the utility u and the deterministic wealth a.* We shall write $H = \mathbf{p_v}(X,a; u)$.

Now we suppose that another economical agent (the purchaser) is interested to purchase the risky asset X. He has the utility U and a deterministic wealth equal to A. If he will purchase X paying a price Π his expected utility would be $\mathrm{E}U(A + X - \Pi)$. As long as $\mathrm{E}U(A + X - \Pi) \geq U(A)$ he will consider the transaction favorable. The maximum price he will be ready to pay for X will be the solution Π of the equation

$$\mathrm{E}U(A + X - \Pi) = U(A) \tag{1.3}$$

The solution Π of equation (1.3) is called the **purchasing price** of X with the utility U and the deterministic wealth A. We shall write $\Pi = \mathbf{p_c}(X,A; U)$.

If the utility is not defined on $\Re$, but on some interval I, some technical problems arise: if we want that equations (1.2) make sense, we should want that $a + X(\omega) \in I\ \forall\ \omega \in \Omega$; and if we want equation (1.3) to make sense, then we should impose the condition that the continuous decreasing function $g(\Pi) = \mathrm{E}U(A + X - \Pi)$ be defined on some interval J such that $\inf\{g(\Pi);\Pi \in J\} \leq U(A)$ and $\sup\{g(\Pi);\Pi \in J\} \geq U(A)$. A necessary condition for the existence of $\mathrm{E}U(A + X - \Pi)$ is that $A + X(\omega) - \Pi \in I\ \forall\ \omega \in \Omega$. For instance, at wealth $A = 0$ and $X \sim \mathrm{Bin}(1,p)$ there is no purchasing price for the utility $u(x) = \sqrt{x}$: equation (1.3) becomes $p\sqrt{1-\Pi} + q\sqrt{-\Pi} = \sqrt{0}$ which has no meaning.

As we intend to focus on other problems, we will avoid such situations. If not stated explicitly otherwise, **our utilities will be functions $u: \Re \to \Re$ defined on the whole real line.**

Now we state the problem. What if the initial wealth A is a random variable ?

Definitions. Let $u: \Re \to \Re$ be a utility function and A, X two random variables such that $u(A+X)$, $u(A)$ and $u(X) \in L^1(\Omega,\mathcal{K},P)$. A is interpreted to be the initial wealth of the decision maker and X the risky asset to be bought or sold . Then any real number H which is a solution of the equation

$$\mathrm{E}u(A+X) = \mathrm{E}u(A+H) \tag{1.4}$$

is called the ***selling price of X at wealth A*** *and is denoted by* $\mathbf{p_v}(X,A;u)$ *or, if no confusion can arise, by* $\mathbf{p_v}(X,A)$. *In the same way a solution* Π *of the equation*

$$\mathrm{E}u(A+X-\Pi) = \mathrm{E}u(A) \tag{1.5}$$

is called the ***purchasing price of X at wealth A*** *and is denoted by* $\mathbf{p_c}(X,A;u)$ *or, if no confusion can arise, by* $\mathbf{p_c}(X,A)$.

The **conditioned selling price of X given A** is a random variable $H(X|A)$ solving the equation

$$\mathrm{E}(u(A+X)|A) = u(A + H(X|A)) \tag{1.6}$$

The **conditioned purchasing price of X given A** is a random variable $\Pi(X|A)$ solving the equation

$$\mathrm{E}(u(A+X - \Pi(X|A))|A) = u(A) \tag{1.7}$$

We shall study the relations between these four concepts.

2. STRAIGHTFORWARD PROPERTIES

Let us start with problems of compatibility.

Definition 2.1. *Let A ,X be two random variables and* $u:I \to \Re$ *a utility function.We say that the pair* (A,X) *is* ***compatible with u*** *if the interval* $(\mathrm{essinf}(A)+\mathrm{essinf}(X), \mathrm{esssup}(A)+\mathrm{esssup}(X))$ *is included in I and* $u(A+X)$, $u(A)$ *and* $u(X)$ *are integrable.*

Let $\alpha = \inf I$ and $\beta = \sup I$. If $\beta = \infty$, the compatibility condition becomes $\alpha \leq \mathrm{essinf}\, A+\mathrm{essinf}\, X$. If, moreover, $\alpha = -\infty$ then there are no compatibility problems.

Proposition 2.2. *Let* u: $I \to \Re$ *be a utility function and* (A,X) *be a pair which is compatible with u. Then the selling price* $\mathbf{p_v}(X,A;u)$ *does exist and is unique.*

Proof. The function $f(H) = Eu(A+H)$, $H \in [\text{essinf}\,(X), \text{esssup}\,(X)]$ is continuous and increasing. Then $f(\text{essinf}\,(X)) = Eu(A + \text{essinf}\, X) \le Eu(A + X) \le Eu(A + \text{esssup}\,(X)) = f(\text{esssup}(X))$. By Darboux's theorem there should exist at least a value H such that $f(H) = Eu(A+X)$. As f is increasing, it is one to one, hence H is unique. This is the selling price $H = \mathbf{p_v}(X,A;u)$. □

Remark. The condition (essinf($A+X$), esssup($A+X$) $\subseteq I$ is not sufficient to ensure the existence of the selling price. Consider for instance $A \sim$ Exponential(λ), $X = -A$, $u(x) = \tan \pi x$. Then $u : (-\frac{1}{2}, \frac{1}{2}) \to \Re$ is a utility. As $A+X = 0$, $u(A+X) = 0$ makes sense, but $u(A+H)$ makes no sense as $A+X$ is not bounded.

As for the purchasing price, its existence is not straightforward. Consider the decreasing function $g(\Pi) = Eu(A+X - \Pi)$. To make sense, we want that $\alpha \le A+X - \Pi \le \beta$ a.s. Otherwise stated, Π is in the domain of g if $\alpha \le \text{essinf}(A+X) - \Pi \le \text{esssup}(A+X) - \Pi \le \beta$ .The condition for the domain to be non-empty is $\alpha - \text{essinf}(A+X) \le \beta - \text{esssup}(A+X) \Leftrightarrow \text{esssup}(A+X) - \text{essinf}(A+X) \le \beta - \alpha$. If the pair (A,X) is compatible with u this is always the case since $\text{esssup}(A+X) - \text{essinf}(A+X) \le \text{esssup}(A)+\text{esssup}(X) - (\text{essinf}(A)+\text{essinf}(X)) \le \beta - \alpha$. But this is not enough.

Proposition 2.3. *Let* u: $I \to \Re$ *be a utility function and* (A,X) *a pair which is compatible with u. If* $Eu(A+X - \text{essinf}(A+X) + \alpha) \ge Eu(A)$ and $Eu(A+X - \text{esssup}(A+X) + \beta) \le Eu(A)$, *then the purchasing price* $\mathbf{p_c}(X,A;u)$ *does exist and is unique.*

Proof. The mapping $g(\Pi) = Eu(A+X - \Pi)$ is decreasing and continuous.□

Example 2.4. Suppose that $X \sim$ Binomial($1,p$), $A = a \ge 0$ is constant and $u(x) = \log x$. Then $Eu(A+X-\Pi) = \frac{1}{2}(\log(a - \Pi) + \log(a+1-\Pi))$, that is well defined for $\Pi < a$. The purchasing price equation (1.5) becomes $(a - \Pi)(a+1-\Pi)) = a^2 \Leftrightarrow \Pi = \sqrt{a^2 + \frac{1}{4}} - \left(a - \frac{1}{2}\right)$. The condition $\Pi < a$ imples $a > \frac{1}{4}$. Plainly, if you have no utility for losses (= negative values) you cannot think of buying something if you are not wealthy enough!

Remark. If the utility u is defined on the whole real line, there are no such problems. The selling and purchasing prices do exist whenever $u(A)$, $u(X)$, $u(A+X)$ are integrable. The most studied utilities with this property are the **CARA utilities**: they are defined by $u(x) = be^{rx} + c$, $r \ne 0$ or by $u(x) = bx+c$ where b,c,r are chosen in such a way that u is increasing. So the CARA utilities are the exponential and linear ones. Usually the prices with nice properties come from CARA utilities (see for example [7], [8]).

In the case of deterministic wealth, the prices given by a utility u are the same with those given by the utility $v = bu + c$ with $b > 0$. This is the ***scale invariance*** of prices with respect to the utility. This property holds in our generalized framework, too. In the same way, the prices are **realistic** in the sense that you cannot pay for a lottery X more than its maximum value or less than its minimum one. These properties are preserved in our generalization.

Proposition 2.5. *Let* u: $I \to \Re$ *be a utility function and* (A,X) *a compatible pair.*

(i). If $m \le X \le M$ (a.s.) *then* $m \le \mathbf{p_v}(X,A;u) \le M$, $m \le \mathbf{p_c}(X,A;u) \le M$

(ii). *If v is another utility, then* $\mathbf{p_v}(X,A;u) = \mathbf{p_v}(X,A;v)$ *can happen for any bounded X,A if an only if v is a scaling of u*: $v = bu + c$ *for some* $b > 0$, $c \in \Re$. *The same holds for purchasing prices.*

Proof.

(i). For the selling price: if $H = \mathbf{p_v}(X,A;u)$ then $Eu(A+m) \le Eu(A+X) = Eu(A+H) \le Eu(A+M) \Rightarrow m \le H \le M$.

For the purchasing one: if $\Pi = \mathbf{p_v}(X,A;u)$ then $Eu(A+X-\Pi) = Eu(A)$. We have $Eu(A + m - \Pi) \le Eu(A+X-\Pi) = Eu(A) \le Eu(A + M - \Pi) \Rightarrow m - \Pi \le 0 \le M - \Pi$.

(ii). If $v = bu + c$ then the equation E$v(A+X)$ = E$v(X+H)$ has the same solution as the equation E$u(A+X)$ = E$u(X+H)$ and the same holds for the purchasing price equation. Conversely, let u and v be two utilities with the property that $\mathbf{p_v}(X,A;u) = \mathbf{p_v}(X,A;v)$ for any bounded X,A. As a particular case, the equality folds for $A = a$ = constant. But then we are in the classic situation: v is a scaling of u (see, for instance, [6] or [14]). □

If u is a utility defined on the whole real line, then the selling price can be computed from

Proposition 2.6. *Suppose that* u: $\Re \to \Re$ *is a utility. Then*

$$\mathbf{p_v}(X,A;u) = \mathbf{p_v}(A+X,0;u) - \mathbf{p_v}(A, \mathbf{p_v}(X,A;u);u) \tag{2.1}$$

$$\mathbf{p_c}(X,A;u) = \mathbf{p_c}(A+X, \mathbf{p_v}(A,0;u);u) - \mathbf{p_v}(A,0;u). \tag{2.2}$$

Thus H, the selling price of X at wealth A, is the difference between the selling price of A+X at wealth 0 *and the selling price of A at wealth H*

Proof. For the selling price equation E$u(A+X)$ = E$u(A + \mathbf{p_v}(X,A, u))$, remark that

$$\mathrm{E}u(A+X) = u(\mathbf{p_v}(A+X,0, u)) \text{ and } \mathrm{E}u(A + \mathbf{p_v}(X,A, u)) = u(\mathbf{p_v}(X,A, u) + \mathbf{p_v}(A, \mathbf{p_v}(X,A, u)). \tag{2.3}$$

Replacing these quantities into it, we get the equation

$$u(\mathbf{p_v}(A+X,0, u)) = u(\mathbf{p_v}(X,A, u) + \mathbf{p_v}(A, \mathbf{p_v}(X,A, u), u). \tag{2.4}$$

Similarly, for the buying price equation E$u(A+X-\Pi)$ = E$u(A)$, replace E$u(A)$ by $u(\mathbf{p_v}(A,0, u))$. Then Π is the solution of a classic buying price equation E$u(A+X-\Pi) = u(\mathbf{p_v}(A,0,u))$.

If we denote $\mathbf{p_v}(A,0, u)$ by a, we can write the above equation as

$$\mathrm{E}u(a +(A+X) - (\Pi+a)) = u(a) \Leftrightarrow \Pi + a = \mathbf{p_c}(A+X, a ;u) \Leftrightarrow \Pi = \mathbf{p_c}(A+X, \mathbf{p_v}(A,0, u), u) - \mathbf{p_v}(A,0, u)$$

And this is exactly equation (2.2)

As a particular case we get

Proposition 2.7 (CARA case). If u is a CARA(r) utility, i.e. if $u(x) = be^{rx} + c$ with $b,c,r \neq 0$ chosen in such a way that u is increasing, then

$$\mathbf{p_v}(X,A;u) = \mathbf{p_c}(X,A;u) = \frac{1}{r}\log\frac{\mathrm{E}e^{r(A+X)}}{\mathrm{E}e^{rA}} = \mathbf{p_v}(A+X,0,u) - \mathbf{p_v}(A,0, u) \tag{2.5}$$

If u is a CARA(0) *utility i.e.* $u(x) = bx + c$, *then* $\mathbf{p_v}(X,A;u) = \mathbf{p_c}(X,A;u) = \mathrm{E}X$

Proof. For CARA utilities both prices are the same and they do not depend on the initial deterministic wealth (see, for instance [14]). But the direct proof is obvious , too: the selling price equation becomes $\mathrm{E}e^{r(A+X)} = e^{rH}\,\mathrm{E}e^{rA}$ and the purchasing price equation is the same $\mathrm{E}e^{r(A+X-\Pi)} = \mathrm{E}e^{rA} \Leftrightarrow e^{rH} = e^{r\Pi} = \dfrac{\mathrm{E}e^{r(A+X)}}{\mathrm{E}e^{rA}}$.

Remark. Equations (2.1) and (2.2) point out that in order to compute the prices $\mathbf{p}(X,A,u)$ we do not need the joint distribution of the vector (A,X), but only the distribution of X,A and $A+X$.

Example 2.8. CARA utility, normal wealth and risky asset. Suppose that u is a CARA utility. Precisely, u may have one of the following three forms:

- $u(x) = ce^{rx} + b$ with $r,c > 0$ or
- $u(x) = b - ce^{rx}$ with $c>0$, $r < 0$ or
- $u(x) = cx+b$ with $c > 0$, if $r = 0$

Then

- $\mathbf{p_v}(X,a; u) = \mathbf{p_c}(X,a; u) = \frac{1}{r}\log \mathrm{E}e^{rX}$ if $r \neq 0$ or
- $\mathbf{p_v}(X,a; u) = \mathbf{p_c}(X,a; u) = \mathrm{E}X$ if $r = 0$

Suppose that the pair (A,X) is normally distributed: $(A,X) \sim N\left(\begin{pmatrix}\mu_A\\ \mu_X\end{pmatrix},\begin{pmatrix}\sigma_A^2 & \rho\sigma_A\sigma_X\\ \rho\sigma_A\sigma_X & \sigma_X^2\end{pmatrix}\right)$. Here ρ is the correlation coefficient between A and X. As $\log Ee^{rA} = r\mu_A + \frac{r^2\sigma_A^2}{2}$, we see that $\mathbf{p_v}(A,0,u) = \mu_A + \frac{r\sigma_A^2}{2}$.

On the other hand, $A+X \sim N(\mu_A + \mu_X, \sqrt{\sigma_A^2 + 2\rho\sigma_A\sigma_X + \sigma_X^2}^{\,2})$, thus

$\mathbf{p_v}(A+X,0) = \mu_A + \mu_X + \frac{r}{2}\left(\sigma_A^2 + 2\rho\sigma_A\sigma_X + \sigma_X^2\right)$.

Applying formula (2.5), we arrive at

$$\mathbf{p_v}(X,A;u) = \mu_A + \mu_X + \frac{r}{2}\left(\sigma_A^2 + 2\rho\sigma_A\sigma_X + \sigma_X^2\right) - (\mu_A + \frac{r\sigma_A^2}{2}) \tag{2.6}$$

hence

$$\mathbf{p_v}(X,A;u) = \mathbf{p_c}(X,A;u) = \mu_X + \frac{r\sigma_X}{2}\left(2\rho\sigma_A + \sigma_X\right) \tag{2.7}$$

Notice that $r = 0 \Rightarrow \mathbf{p_c}(X,A;u) = \mu_X$, as it should be: for linear utilities, the price is equal to the expectation.

A mysterious fact arises. We know (see [14] for instance) that when the initial wealth is deterministic and the utility is concave, both prices are smaller than the expectation. Not in the general case. For instance, if $r = -2$ and $\rho = -½$, we get $\mathbf{p_v}(X,A;u) = \mathbf{p_c}(X,A;u) = \mu_X - \sigma_X(\sigma_X - \sigma_A)$. If $\sigma_A > \sigma_X$ we see that $\mathbf{p_v}(X,A;u) > \mu_X$!

Searching for an explanation, we compute the conditioned selling price defined by equation (1.6). The conditioned distribution of X given A is a normal distribution, too (see, for instance [3] or [4]) : precisely

$(X \mid A) \sim N\left(\mu_X + \rho\frac{\sigma_X}{\sigma_A}(A-\mu_A), \left(\sigma_X\sqrt{1-\rho^2}\right)^2\right)$.

It follows that

$$H(X \mid A) = \mu_X + \rho\frac{\sigma_X}{\sigma_A}(A-\mu_A) + \frac{r\sigma_X^2}{2}\left(1-\rho^2\right). \tag{2.8}$$

It is true that $r \le 0 \Rightarrow H(X \mid A) \le E(X \mid A) \Rightarrow E[H(X \mid A)] \le EX = \mu_X$, but the point is that the inequality $\mathbf{p_v}(X,A;u) \le \mu_X$ does not hold anymore. As $H(X \mid A) \sim N(\mu_X + \frac{r\sigma_X^2}{2}\left(1-\rho^2\right), \rho^2\sigma_X^2)$, we have

$$E[H(X \mid A)] = \mu_X + \frac{r\sigma_X^2}{2}\left(1-\rho^2\right) \tag{2.9}$$

and remark that there is no obvious relation between (2.9) and (2.7). Notice however that if A and X are not correlated, then $H(X \mid A) = \mathbf{p_v}(X,0) = \mu_X + \frac{r\sigma_X^2}{2}$.

3. ORDER PRESERVING PROPERTIES OF A TRANSLATION TYPE OPERATOR

Let A be a random variable. Denote by $\mathcal{U} = \{ u:\mathbb{R} \to \mathbb{R}: u \text{ is a utility function}\}$. Define the translation operator $T_A : \mathcal{U} \to \mathcal{U}$, $T_A(u) = u_A$ where the new utility u_A is defined by

$$u_A(x) = Eu(A + x). \tag{3.1}$$

If the distribution of A is F, we can write

$$u_A(x) = \int u(a+x)\mathrm{d}F(a)\,. \tag{3.2}$$

According to (1.4) the selling price $\mathbf{p_v}(X,A;u)$ is the solution H of the equation

$$\mathrm{E}u(A+X) = \mathrm{E}u(A+H) \quad \text{or} \quad \int u(a+x)\mathrm{d}F_{A,X}(a,x) = \int u(a+H)\mathrm{d}F(a)\,, \tag{3.3}$$

where $F_{A,X}$ is the distribution of the pair (A,X). In the particular case when A and X are independent, then $F_{A,X} = F_A \otimes F_X$, where F_X is the distribution of X. In this particular case equation (3.3) can be written as

$$\int \left(\int u(a+x)\mathrm{d}F(a)\right)\mathrm{d}F_X(x) = \int u(a+H)\mathrm{d}F(a) \quad \text{or} \quad \mathrm{E}u_A(X) = u_A(H) \tag{3.4}$$

But this is the equation for the selling price $\mathbf{p_v}(X,0;u_A)$!. We arrive at

Proposition 3.1. If X and A are independent, then

$$\mathbf{p_v}(X,A;u) = \mathbf{p_v}(X,0;u_A) \;\; \textit{and} \;\; \mathbf{p_c}(X,A;u) = \mathbf{p_c}(X,0;u_A) \tag{3.5}$$

Proof. The first relation is already proved. For the second one, notice that $\mathrm{E}u(X+A-\Pi) = \mathrm{E}u_A(X-\Pi)$ and $\mathrm{E}u(A) = u_A(0)$, hence the buying price equation $\mathrm{E}u(X+A-\Pi) = \mathrm{E}u(A)$ becomes $\mathrm{E}u_A(X - \Pi) = u_A(0)$; this is an equation whose solution is $\Pi = \mathbf{p_c}(X,0;u_A)$.

In [14] we considered the relation

$$u \prec v \iff \mathbf{p_v}(X,a;u) \le \mathbf{p_v}(X,a;v) \quad \forall\, a,X \tag{3.6}$$

between two utilities and showed that if u and v are twice differentiable utilities, then $u \prec v$ is equivalent to $r_u \le r_v$ where $r_u = \dfrac{u''}{u'}$ and $r_v = \dfrac{v''}{v'}$ are the so called ***risk-seeking coefficients*** *of u and v*.

If we may commute expectation and differentiation, then we obtain formulae for the risk seeking coefficients of u_A and v_A:

Proposition 3.2. Suppose that u_A and v_A have the property

$(u_A)'(x) = \mathrm{E}u'(A+x)$, $(u_A)''(x) = \mathrm{E}u''(A+x)$, $(v_A)'(x) = \mathrm{E}v'(A+x)$, $(v_A)''(x) = \mathrm{E}v''(A+x)$

Then $r_{u_A}(x) = \dfrac{\mathrm{E}u''(A+x)}{\mathrm{E}u'(A+x)},\; r_{v_A}(x) = \dfrac{\mathrm{E}v''(A+x)}{\mathrm{E}v'(A+x)}$

As a consequence

$$r_{u_A} \prec r_{v_A} \iff \mathrm{E}u''(A+x)\mathrm{E}v'(A+x) \le \mathrm{E}v''(A+x)\mathrm{E}u'(A+x) \tag{3.7}$$

One may see that the relation $u \prec v$ does not necessarily imply $u_A \prec v_A$.

Counter-example. Suppose that $A \sim \begin{pmatrix} 0 & a \\ 1-\frac{1}{a} & \frac{1}{a} \end{pmatrix}$ for $a > 1$. Notice that $\mathrm{E}A = 1$, $\mathrm{E}A^2 = a$.

Let $u(x) = x^2$ and $v(x)=x^3$. Both u and v are utilities on the interval $(0,\infty)$. As $r_u(x) = \dfrac{1}{x}$ and $r_v(x) = \dfrac{2}{x}$, we have $u \prec v$. This means that the selling price given by u is always smaller than the selling price given by *v, at any initial wealth a*.

Now, $u_A(x) = \left(1-\dfrac{1}{a}\right)x^2 + \dfrac{1}{a}(a+x)^2 = x^2 + 2x + a \Rightarrow r_{u_A}(x) = \dfrac{1}{x+1}$

and $v_A(x) = \left(1-\dfrac{1}{a}\right)x^3 + \dfrac{1}{a}(a+x)^3 = x^3 + 3x^2 + 3ax + a^2 \Rightarrow r_{v_A}(x) = \dfrac{2(x+1)}{x^2+2x+a}$.

The inequality $r_{u_A} \le r_{v_A}$ is equivalent to $x^2 + 2x + a \le 2(x+1)^2 \Leftrightarrow x \ge \sqrt{a-1} - 1$. For instance, for $a = 10$, $r_{u_A} \le r_{v_A} \Leftrightarrow x \ge 2$.

Now, suppose that X and A are independent and $X \sim$ Uniform(0,1). Let $H_u = \mathbf{p_v}(X,A;u)$ and

$H_v = \mathbf{p_v}(X,A;v)$. Curiously, $H_u = \sqrt{\frac{7}{3}} - 1 = .52752\ldots$ does not depend on a; H_v is the solution s of the equation $\mathrm{E}(A+X)^3 = \mathrm{E}(A+s)^3 \Leftrightarrow s^3 + 3s^2\mathrm{E}A + 3s\mathrm{E}A^2 + \mathrm{E}A^3 = \mathrm{E}X^3 + 3\mathrm{E}X^2\mathrm{E}A + 3\mathrm{E}X\mathrm{E}A^2 + \mathrm{E}A^3 \Leftrightarrow s^3 + 3s^2 + 3as = \frac{1}{4} + 1 + 3a/2$.

For $a = 10$ the equation becomes $s^3 + 3s^2 + 30s - 16.25 = 0$. Its solution is $s = 0.5111\ldots$.

Thus, it is not true that $u_A \prec v_A$ if $u \prec v$ even for utilities of the form $u(x) = x^p$.

However, strangely enough, the assertion holds for utilities of the form $u(x) = x^p$ *for* $0 \le p \le 1$. Here is the precise result

Proposition 3.3. *Suppose* $0 \le p < q \le 1$ and $u(x) = x^p$, $v(x) = x^q$. *If* $p = 0$ *we take* $u(x) = \log x$. *Then* $u_A \prec v_A$ *for every random wealth* $A \ge 0$, (*not equal to* 0 *almost surely*) *for which* $\mathrm{E}A^q < \infty$.

Proof. Let $x, y \ge 0$. Since $(x+y)^p \le x^p + y^p$, both $u_A(x) = \mathrm{E}(A+x)^p$ and $v_A(x) = \mathrm{E}(A+x)^q$ are finite. It is easy to see that we can commute differentiation and expectation, (use Lebesgue's dominated convergence theorem). We are going to check that equation (3.7) holds and that will complete the proof. Let $B = A+x$. Let us check that $\mathrm{E}u''(B)\mathrm{E}v'(B) \le \mathrm{E}v''(B)\mathrm{E}u'(B)$. This inequality is equivalent to

$(q-1)\,\mathrm{E}B^{q-2}\mathrm{E}B^{p-1} - (p-1)\,\mathrm{E}B^{p-2}\mathrm{E}B^{q-1} \ge 0.$

Let F be the distribution of B. Then

$$(q-1)\,\mathrm{E}B^{q-2}\mathrm{E}B^{p-1} = (q-1)\iint x^{q-2}y^{p-1}\mathrm{d}F(x)\mathrm{d}F(y)$$

Exchanging x and y we can write

$$2(q-1)\,\mathrm{E}B^{q-2}\mathrm{E}B^{p-1} = \iint (q-1)(x^{q-2}y^{p-1} + y^{q-2}x^{p-1})\mathrm{d}F(x)\mathrm{d}F(y)$$

and

$$2(p-1)\,\mathrm{E}B^{p-2}\mathrm{E}B^{q-1} = \iint (p-1)(x^{p-2}y^{q-1} + y^{p-2}x^{q-1})\mathrm{d}F(x)\mathrm{d}F(y).$$

Denote $\phi(x,y) = (q-1)(x^{q-2}y^{p-1} + y^{q-2}x^{p-1}) - (p-1)(x^{p-2}y^{q-1} + y^{p-2}x^{q-1})$, $x, y \ge 0$. One can easily see that

$$2(q-1)\mathrm{E}B^{q-2}\mathrm{E}B^{p-1} - 2(p-1)\mathrm{E}B^{p-2}\mathrm{E}B^{q-1} = \iint \phi(x,y)\mathrm{d}F(x)\mathrm{d}F(y).$$

We shall check that $\phi(x,y) \ge 0$. This is equivalent to

$$(1-q)(x^{q-2}y^{p-1} + y^{q-2}x^{p-1}) - (1-p)(x^{p-2}y^{q-1} + y^{p-2}x^{q-1}) \le 0 \ \forall\, x,y > 0.$$

As $1-q \le 1-p$,

$(1-q)(x^{q-2}y^{p-1} + y^{q-2}x^{p-1}) - (1-p)(x^{p-2}y^{q-1} + y^{p-2}x^{q-1}) \le (1-q)(x^{q-2}y^{p-1} + y^{q-2}x^{p-1}) - (1-q)(x^{p-2}y^{q-1} + y^{p-2}x^{q-1})$.

It is enough to show that

$$(x^{q-2}y^{p-1} + y^{q-2}x^{p-1}) - (x^{p-2}y^{q-1} + y^{p-2}x^{q-1} \le 0.$$

Replacing y by tx, $t > 0$ the claimed inequality becomes

$$t^{p-1} + t^{q-2} - t^{q-1} - t^{p-2} \le 0 \Leftrightarrow t^{p-2}(t-1) + t^{q-2}(1-t) \le 0 \Leftrightarrow (t^{p-2} - t^{q-2})(t-1) \le 0$$

and that is obvious.

4. ACKNOWLEDGEMENTS

The authors acknowledge financial support from the CEEX – National Research and Development Program of the Ministry of Education and Research –Contracts 28/2005 and 26/2006 and from the CNCSIS contracts 1591/2003 and 831/2004.

5. REFERENCES LIST

1. ANAND, P., *Foundations of Rational Choice Under Risk.* Oxford University Press, Oxford 2002.
2. BERGER, J. O, *Statistical Decision Theory and Bayesian Analysis.* Springer, Berlin 1985.
3. CIUCU, G., TUDOR, C., *Probability theory,* Ed. Academiei, Bucharest 1981 (in Romanian).
4. CUCULESCU, I., *Probability Theory.* ALL, Bucharest 1998, (in Romanian).
5. DOHERTY, Neil A., SCHLESINGER, H., *A note on risk premiums with random initial wealth*, Insurance: Mathematics and Economics, 5, Issue 3, 183-185, 1986.
6. FOLLMER, H., SCHIED, A., Stochastic Finance. An Introduction to Discrete Time. De Gruyter, Berlin, 2002.
7. FRIEDMAN, M., SAVAGE, L. J., The *Utility analysis of choices involving risk,* The Journal of Political Economy, 56, No. 4, 279-304, 1948.
8. KIHLSTROM, R. E., ROMER D., WILLIAMS S., *Risk Aversion with Random Initial Wealth*, Econometrica, 49, No. 4, 911-920, 1981.
9. MAHUL, O., *Optimal insurance design with random initial wealth*, *Economics Letters* 69, 3, 353-358, 2000.
10. MOCIOALCA, O., *Jump diffusion options with transaction costs*, Rev. Roumaine Math. Pures Appl., 52, 3, 349-366, 2007.
11. NEUMANN, J., MORGENSTERN, O. *Theory of Games and Economic Behavior,* NJ Princeton University Press, Princeton, 1947.
12. ZBĂGANU, G., RĂDULESCU, M., *Properties of selling and buying prices.* Proc. 9th WSEAS Int. Conf. on Mathematics and Computers in Business and Economics (MCBE'08), Bucharest 2008, pg. 44-49. WSEAS Press. 2008.
13. ZBĂGANU, G., RĂDULESCU, M., *Existence Conditions for Trading Transactions.* Proc. 12th WSEAS Int. Conf. on Computers, Heraklion, Greece, 2008, (2008), pg. 1097-1100. WSEAS Press 2008.
14. ZBĂGANU, G., *Mathematical Methods in Risk Theory and Actuaries.* Bucharest, Ed. Univ. Bucharest 2004 (in Romanian).

Received October 1, 2008