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THE UNITS IN $\widetilde{\mathbb{Q}}_p$

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Let p be a prime number. The ring $\widetilde{\mathbb{Q}}_p$ has been defined in [6] as being the completion of the field $\overline{\mathbb{Q}}$ of algebraic numbers with respect to the so called spectral norm. Since this ring seems to be interesting from many points of view, we want to continue to investigate its properties of it. The aim of this paper is to investigate the units and the regular polynomials of $\widetilde{\mathbb{Q}}_p$.

Key words: pseudovaluation, spectral extensions, "Krull topology", spectral norm, regular polynomial

1. INTRODUCTION

In [6] it is defined the ring $\widetilde{\mathbb{Q}}_p$. This paper is a natural continuation of our previous papers [2], [6]. We intend here to describe the units sets and the regular polynomials of $\widetilde{\mathbb{Q}}_p$.

The paper is divided into four sections. The second section presents the notations, definitions and basis re-sults. In Section 3, the units in $\widetilde{\mathbb{Q}}_p$ are studied. Finally, in Section 4 we study the regular polynomials in $\widetilde{\mathbb{Q}}_p$.

2. NOTATIONS, DEFINITIONS AND BASIC RESULTS

Denote by $\mathbb{Q}$ the field of rational numbers and by $\overline{\mathbb{Q}}$ a fixed closure of it. Let p be a prime number and let v_p denote the valuation on $\overline{\mathbb{Q}}$ defined by it [1]. Also denote by v a fixed valuation on $\overline{\mathbb{Q}}$ which extends v_p .

Denote by $G = Gal\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ the Galois group of automorphism of $\overline{\mathbb{Q}}$. For any $\sigma \in G$, denote by σv the valuation on $\overline{\mathbb{Q}}$ defined by $(\sigma v)(x) = v(\sigma^{-1}(x))$ for all $x \in \overline{\mathbb{Q}}$. Then σv is also an extension of v_p to $\overline{\mathbb{Q}}$ and any valuation v' on $\overline{\mathbb{Q}}$ which extends v_p is of this form.

Let $\mathbb{Q}_p$ be the field of p -adic numbers and let us continue to denote by v_p the unique valuation on $\mathbb{Q}_p$ which extends v_p . Also let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$, and continue to denote by v_p the unique valuation on $\overline{\mathbb{Q}}_p$ which extends v_p .

Finally, let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}}_p$ with respect to v_p and again continue to denote by v_p the unique valuation on $\mathbb{C}_p$ which extends v_p . The valued field $\left(\mathbb{C}_p, v_p\right)$ is called the complex p -adic field.

Consider the valued field $(\mathbb{Q},v_p)$. Denote by w the spectral pseudovaluation on $\overline{\mathbb{Q}}$ induced by the valuation v_p on $\mathbb{Q}$.

According to [6] for any $x\in\overline{\mathbb{Q}}$ one has $w(x)=\inf_{\sigma\in G}v\left(\sigma^{-1}(x)\right)=\inf_{\sigma\in G}(\sigma v)(x).$

We shall say that w is the spectral extension of v_p to $\overline{\mathbb{Q}}$.

Denote by $\widetilde{\mathbb{Q}}_p$ the completion of $\overline{\mathbb{Q}}$ with respect to spectral pseudovaluation w above defined. Also, denote by $\tilde{w}$ the unique extensions of w to $\widetilde{\mathbb{Q}}_p$. Since w is not a valuation, then $\widetilde{\mathbb{Q}}_p$ is not an integral domain ([6] Corollary 2.5.).

Denote by $\|\cdot\|$ the spectral norm on $\widetilde{\mathbb{Q}}_p$ which corresponds to w.

According to [6], for any $x\in\widetilde{\mathbb{Q}}_p$ one has $\|x\|=\left(\frac{1}{p}\right)^{\tilde{w}(x)}$.

Let $x\in\widetilde{\mathbb{Q}}_p$. For any $\sigma\in G$, we define the map $\theta_\sigma:\widetilde{\mathbb{Q}}_p\to\mathbb{C}_p$ in the following way.

Let $x=\{x_n\}_n$, where $\{x_n\}_n$, is a Cauchy sequence in the class of x, $x_n\in\overline{\mathbb{Q}}$ for any $n=1,2,\ldots$. Then $\{\sigma(x_n)\}_n$ is a Cauchy sequence in $(\mathbb{C}_p,v_p)$. Let us put $\theta_\sigma(x)$ is class there. Shortly we write $x_\sigma=\theta_\sigma(x)$.

The element $x\in\widetilde{\mathbb{Q}}_p$ is well defined by its components $\{x_\sigma\}_{\sigma\in G}$.

In fact we have $\|x\|=\sup_{\sigma\in G}(|x_\sigma|)$, where $|x_\sigma|=|x_\sigma|_p=\left(\frac{1}{p}\right)^{v_p(x_\sigma)}$ is the absolute value of x_σ in $\mathbb{C}_p$.

If $x\in\widetilde{\mathbb{Q}}_p$, then $x=\lim_w x_n$, where $\{x_n\}_n$ is a Cauchy sequence of $\overline{\mathbb{Q}}$ with respect to w. This means that $w(x_{n+1}-x_n)=\inf_{\tau\in G}(\tau v)(x_{n+1}-x_n)\xrightarrow[n\to\infty]{}+\infty.$

If $x,a\in\widetilde{\mathbb{Q}}_p$, $x=(x_\sigma)_{\sigma\in G}$, $a=(a_\sigma)_{\sigma\in G}$ and if $r>0$ is a real number, we mark with $\tilde{B}=\tilde{B}(a,r)=\left\{x\in\widetilde{\mathbb{Q}}_p \,\middle|\, \|x-a\|<r\right\}$ the open ball with the center a and the radius r, and with $\tilde{B}'=\tilde{B}'(a,r)=\left\{x\in\widetilde{\mathbb{Q}}_p \,\middle|\, \|x-a\|\le r\right\}$ the closed ball with the center a and the radius r.

Denote with $B(a_\sigma,\varepsilon)=\left\{x_\sigma\in\mathbb{C}_p \,\middle|\, |x_\sigma-a_\sigma|<r\right\}$ the open ball with the center a and the radius r, and

$B'(a_\sigma,\varepsilon)=\left\{x_\sigma\in\mathbb{C}_p \,\middle|\, |x_\sigma-a_\sigma|\le r\right\}$ the closed ball with the center a_σ and the radius r.

Let us consider $x\in\widetilde{\mathbb{Q}}_p$, $x=(x_\sigma)_{\sigma\in G}$ and $\rho>0$ a real number. If for any $\sigma\in G$ we have $|x_\sigma|<\rho$, then $\|x\|<\rho$ ([3] Lemma 2.1.). The set $C_1(x)=\{x_\sigma \mid \sigma\in G\}$ is compact ([2] Remarks 3.9.).

Let $y\in\widetilde{\mathbb{Q}}_p$ and $\varepsilon>0$ be a real number. Then there exists an element $z\in\overline{\mathbb{Q}}$ with the following properties:

i) $|y-z|\le\varepsilon$;

ii) $\deg_{\mathbb{Q}}(z)=\deg_{\mathbb{Q}_p}(z)=\deg_{\mathbb{Q}_p}(y)$ ([2] Proposition 4.1.).

3. THE UNITS IN $\widetilde{\mathbb{Q}}_p$

1. Denote by U_p the group of units of $\tilde{\mathbb{Q}}_p$, that is $U_p=\left\{x\in\tilde{\mathbb{Q}}_p \,\middle|\, (\exists)y\in\tilde{\mathbb{Q}}_p \textit{ so that } xy=1\right\}$.

We have the following theorem of characterization of the inversibles elements of the ring $\tilde{\mathbb{Q}}_p$.

Theorem 3.1 *Let us consider* $x=\left(x_\sigma\right)_{\sigma\in G}$ *from* $\tilde{\mathbb{Q}}_p$. *The element* x *is inversible if and only if* $x_\sigma \neq 0$, *for any* $\sigma \in G$.

Proof. Let the element $x\in\tilde{\mathbb{Q}}_p$ be inversible. Then there is $y=\left(y_\sigma\right)_{\sigma\in G}$, $y\in\tilde{\mathbb{Q}}_p$ so that $xy=1$. Therefore $\left(x_\sigma\right)\left(y_\sigma\right)=(xy)_\sigma=1$, that is $x_\sigma\cdot y_\sigma=1$, for any $\sigma\in G$, which shows that $x_\sigma\neq 0$ for any $\sigma\in G$.

Conversely, consider $x\in\tilde{\mathbb{Q}}_p$, $x=\left(x_\sigma\right)_{\sigma\in G}$, so that $x_\sigma\neq 0$, for any $\sigma\in G$. We shall prove that there exists $x^{-1}\in\tilde{\mathbb{Q}}_p$.

Consider the set $C_1(x)=\left\{x_\sigma \middle| \sigma\in G\right\}$ from $\mathbb{C}_p$. Using Remarks 3.9. of [2] we have that the set $C_1(x)$ is compact and closed, and the zero element does not belong to $C_1(x)$. We state that there exists a real number $\varepsilon>0$, so that $B(0,\varepsilon)\cap C_1(x)=\varnothing$.

Indeed, if for any $\varepsilon>0$ there is an element $a_\varepsilon\in C_1(x)\cap B(0,\varepsilon)$, then we infer that there exists a sequence $\{a_n\}_n\subseteq C_1(x)$ convergent to zero.

But in this case the zero element is in set $C_1(x)$, which is a contradiction.

So that let us consider ε_0 a real number with $B(0,\varepsilon_0)\cap C_1(x)=\varnothing$.

Then for any $\sigma\in G$ we have $\left|x_\sigma\right|\geq\varepsilon_0$ and so $\frac{1}{\left|x_\sigma\right|}\leq\frac{1}{\varepsilon_0}$. Let $\left\{x_n\right\}_n$ be a sequence of elements in $\overline{\mathbb{Q}}$ with $\lim\limits_{\|\cdot\|}x_n=x$.

We know that for n large enough we have $\|x_n\|=\|x\|$, so we have $\|x_n\|=\sup\limits_{\sigma\in G}\left(\left|\left(x_n\right)_\sigma\right|\right)=\|x\|=\sup\limits_{\sigma\in G}\left(|x_\sigma|\right)$.

Consider $\delta>0$ be a small real number. Then for n large enough we have $\left\|x-x_n\right\|<\delta$, or $\sup\limits_{\sigma\in G}\left(\left|x_\sigma-\left(x_n\right)_\sigma\right|\right)<\delta$, we have $\left|x_\sigma-\left(x_n\right)_\sigma\right|<\delta$, for any $\sigma\in G$.

We can consider $\delta<\varepsilon_0$, and then we have $\left|\left(x_n\right)_\sigma\right|=\left|\left(x_n\right)_\sigma-x_\sigma+x_\sigma\right|=\left|x_\sigma\right|$, where $\left|x_\sigma\right|\geq\varepsilon_0$.

So we obtained $\left|\left(x_n\right)_\sigma\right|\geq\varepsilon_0$, for any $\sigma\in G$ and n large enough.

Let us consider the sequence $\left\{y_n\right\}_n$, where $y_n=\frac{1}{x_n}$. We shall prove that this sequence has a limit.

For n large enough, since $\left|\left(x_n\right)_\sigma\right|\geq\varepsilon$ we have

$$\left|\left(y_{n+1}\right)_\sigma-\left(y_n\right)_\sigma\right|=\left|\frac{1}{\left(x_{n+1}\right)_\sigma}-\frac{1}{\left(x_n\right)_\sigma}\right|=\left|\left(\frac{x_n-x_{n+1}}{x_n\cdot x_{n+1}}\right)_\sigma\right|$$

$$=\frac{\left|\left(x_n-x_{n+1}\right)_\sigma\right|}{\left|\left(x_n\right)_\sigma\right|\cdot\left|\left(x_{n+1}\right)_\sigma\right|}\leq\frac{\left|\left(x_n-x_{n+1}\right)_\sigma\right|}{\varepsilon_0^2}$$

and so

$$\sup_{\sigma\in G}\left|\left(\frac{1}{x_{n+1}}-\frac{1}{x_n}\right)_\sigma\right|\leq\frac{\sup\limits_{\sigma\in G}\left|\left(x_n-x_{n+1}\right)_\sigma\right|}{\varepsilon_0^2}.$$

Hence, the sequence $\left\{\frac{1}{x_n}\right\}_n$ is convergent in the spectral norm.

Let $y=\lim\limits_{\|\cdot\|}\frac{1}{x_n}$. Then we have $xy=\lim\limits_{\|\cdot\|}\left(x_n\cdot\frac{1}{x_n}\right)=\lim\limits_{\|\cdot\|}1=1$, that is $y=x^{-1}$, $y\in\tilde{\mathbb{Q}}_p$.

In order to render evident both the topological and the algebraic aspects of the problem of units in $\tilde{\mathbb{Q}}_p$, we shall give two proofs for the following theorem.

Theorem 3.2 *If* $x \in \tilde{\mathbb{Q}}_p$ *and* $\| x \| < 1$*, then the element* $1 - x$ *is inversible.*

Proof. The first proof of this theorem is based onTheorem 3.1. Indeed, if $\| x \| < 1$ then $\| x \| = \sup_{\sigma \in G} (|x_\sigma|) < 1$ and so $|x_\sigma| < 1$, for any $\sigma \in G$. But then $|1 - x_\sigma| = 1$, for any $\sigma \in G$ which proves that $1 - x_\sigma \neq 0$, for any $\sigma \in G$ and so the element $1 - x$ is inversible.

The second proof is classical and is based on the fact that, if $\| x \| < 1$, then the series $1 + x + x^2 + \ldots + x^n + \ldots$ is convergent.

Let A be the sum of this series.

We have $A = \lim_{n \to \infty} \left(1 + x + x^2 + \ldots + x^n\right)$, but then $A(1-x) = \lim_{n \to \infty} (1-x)\left(1 + x + x^2 + \ldots + x^n\right) = \lim_{n \to \infty} \left(1 - x^{n+1}\right) = 1$ and so $A = \dfrac{1}{1-x}$.

2. The problem now is to find out how many units there are in $\tilde{\mathbb{Q}}_p$. Denote by $\theta_\sigma : U_p \to \mathbb{C}_p^{\star}$, $\theta_\sigma(x) = x_\sigma$ where $\mathbb{C}_p^{\star} = \mathbb{C}_p \setminus \{0\}$.

Theorem 3.3 *The function* $\theta_\sigma : U_p \to \mathbb{C}_p^{\star}$ *defined by* $\theta_\sigma(x) = x_\sigma$ *is surjective.*

Proof. Let $z \in \mathbb{C}_p^{\star}$ and $\sigma \in G$.

We shall prove that there is an element $x \in U_p$ so that $x_\sigma = z$.

For this we shall use Proposition 4.1. of [2]. So consider the sequence $\{z_n\}_n \in \overline{\mathbb{Q}}_p$ so that $\lim_{\|\cdot\|} z_n = z$. For any $n \geq 1$, we choose an element $x_n \in \overline{\mathbb{Q}}$, according to Proposition 4.1. of [2] so that $|x_n - z_n| < \dfrac{1}{n}$ and $\deg_{\mathbb{Q}_p}(x_n) = deg_{\mathbb{Q}}(x_n) = deg_{\mathbb{Q}_p}(z_n)$.

We state that the sequence $\{x_n\}_n$ converges to z.

Indeed

$$|x_{n+1} - x_n| = |x_{n+1} - z_{n+1} + z_{n+1} - z_n + z_n - x_n| \leq \max\left(|x_{n+1} - z_{n+1}|, |z_{n+1} - z_n|, |z_n - x_n|\right)$$

$$\leq \max\left(\frac{1}{n}, \frac{1}{n+1}, |z_{n+1} - z_n|\right).$$

Since $\lim_{\|\cdot\|} \left(z_{n+1} - z_n\right) = 0$ it is inferred that $\lim_{\|\cdot\|} \left(x_{n+1} - x_n\right) = 0$ and so the sequence $\{x_n\}_n$ is convergent in the p-adic norm. But, according to Proposition 4.1. of [2] it can be inferred that the sequence $\{x_n\}_n$ is also convergent in the spectral norm, since $v\left(x_{n+1} - x_n\right) = v\left(\sigma\left(x_{n+1}\right) - \sigma\left(x_n\right)\right)$, for any $\sigma \in G$ [6].

Hence $\lim_{\|\cdot\|} x_n = x_\sigma = z$, for any $\sigma \in G$ and thus we have proved that the function θ_σ is surjective. □

Remark 3.4 *Let* $x \in U_p$*. We have* $\| x \| = \sup_{\sigma \in G} |x_\sigma|$ *and since* $x^{-1} = \left(x_\sigma^{-1}\right)_{\sigma \in G}$*, we obtain*

$$\left\| x^{-1} \right\| = \sup_{\sigma \in G} \left(\left| \frac{1}{x_\sigma} \right| \right) = \frac{1}{\inf_{\sigma \in G} \left(|x_\sigma| \right)}.$$

Since the sets $C_1(x) = \{x_\sigma | \sigma \in G\}$ and $C_1\left(x^{-1}\right) = \{x_\sigma^{-1} | \sigma \in G\}$ are closed and compact, it is immediately inferred that $\| x \| = \sup_{\sigma \in G} \left(|x_\sigma|\right) \in |C_1(x)|$ as well as that $\left\| x^{-1} \right\| = \dfrac{1}{\inf_{\sigma \in G} |x_\sigma|} \in \left|C_1\left(x^{-1}\right)\right|$, where $| C_1(x) | = \{ | x_\sigma | \,|\, \sigma \in G \}$.

Now let $\left|x_{\sigma_0}\right| = \| x \| = \sup_{\sigma\in G}\left|x_\sigma\right|$ and $\left|x_{\sigma_1}^{-1}\right| = \left\|x^{-1}\right\| = \dfrac{1}{\inf_{\sigma\in G}\left(\left|x_\sigma\right|\right)}$. But $\left|x_{\sigma_0}^{-1}\right| \le \left\|x^{-1}\right\| = \dfrac{1}{\left|x_{\sigma_1}\right|}$ or $\left|x_{\sigma_0}^{-1}\right| \le \left|x_{\sigma_1}^{-1}\right|$.

We state that $\left|x_{\sigma_0}^{-1}\right| = \left|x_{\sigma_1}^{-1}\right|$. Indeed, we suppose that

$$\left|x_{\sigma_0}^{-1}\right| < \left|x_{\sigma_1}^{-1}\right|. \qquad (1)$$

But we have $\left|x_{\sigma_0} x_{\sigma_0}^{-1}\right| = 1 = \left|x_{\sigma_0}\right| \cdot \left|x_{\sigma_0}^{-1}\right| < \left|x_{\sigma_0}\right| \cdot \left|x_{\sigma_1}^{-1}\right| \le \left|x_{\sigma_1}\right| \cdot \left|x_{\sigma_1}^{-1}\right| = 1$, because $\left|x_{\sigma_1}\right| = \| x \| \ge \left|x_{\sigma_0}\right|$.

Thus we have obtained a contradiction and this shows that the inequality (1) is not true. So $\left|x_{\sigma_0}^{-1}\right| = \| x \|^{-1} = \left|x_{\sigma_1}^{-1}\right| = \left\|x^{-1}\right\|$, that is $\left\|x^{-1}\right\| = \left\|x\right\|^{-1}$, or $\left\|x\right\| \cdot \left\|x^{-1}\right\| = 1$.

3. Denote by $I(a) = \left\{ x \in \tilde{\mathbb{Q}}_p \middle| x - a \in U_p \right\}$.

Now we intend to examine the points of continuity for the function $f : I(a) \to \tilde{\mathbb{Q}}_p$ defined by $f(x) = \dfrac{1}{x-a}$, where $x \not\asymp a$. We have $x - a \in U_p$ if and only if $x_\sigma - a_\sigma \not\asymp 0$, for any $\sigma \in G$.

So $| x_\sigma - a_\sigma | > 0$, for any $\sigma \in G$.

Proposition 3.5 *If* $x \in I(a)$, *then* $\inf_{\sigma\in G}\left(\left|x_\sigma - a_\sigma\right|\right) > 0$.

Proof. We suppose that $\inf_{\sigma\in G}\left(\left|x_\sigma - a_\sigma\right|\right) = 0$.

Since G is a compact space in the "Krull topology", any sequence has a limit point.

Consider the sequence $\left\{\sigma_n\right\}_n \subset G$ so that $\lim_{|\cdot|}\left(x_{\sigma_n} - a_{\sigma_n}\right) = 0$.

Since $\left\{x_{\sigma_n}\right\}_n \subset C_1(x)$ and $\left\{a_{\sigma_n}\right\}_n \subset C_1(a)$, and the set $C_1(x)$ and $C_1(a)$ are closed sets, we infer that from the sequences $\left\{x_{\sigma_n}\right\}_n$ and $\left\{a_{\sigma_n}\right\}_n$ we can extract some convergent subsequences $\left\{x_{\bar{\sigma}_n}\right\}_n \subseteq \left\{x_{\sigma_n}\right\}_n$ and $\left\{a_{\bar{\sigma}_n}\right\}_n \subseteq \left\{a_{\sigma_n}\right\}_n$.

We suppose that

$$\lim_{n\to\infty} x_{\bar{\sigma}_n} = \bar{x} = x_t \text{ and } \lim_{n\to\infty} a_{\bar{\sigma}_n} = \bar{a} = a_t. \qquad (2)$$

Let $\sigma \in G$, so that

$$\lim_{n\to\infty} \sigma_n = \sigma. \qquad (3)$$

Since $\lim_{|\cdot|}\left(x_{\sigma_n} - a_{\sigma_n}\right) = 0$, and taking into account the relations (2) and (3), we infer that $\bar{x} = x_\sigma$, $\bar{a} = a_\sigma$, that is $a_\sigma = x_\sigma$. Thus we have obtained a contradiction.

Therefore, if $x \in I(a)$, then $\inf_{\sigma\in G}\left(\left|x_\sigma - a_\sigma\right|\right) > 0$.

Proposition 3.6 *The set* $I(a)$ *is an open set.*

Proof. Let $x \in I(a)$, that is $\inf_{\sigma\in G}\left(\left|x_\sigma - a_\sigma\right|\right) > 0$. Denote by $d = \inf_{\sigma\in G}\left(\left|x_\sigma - a_\sigma\right|\right)$. Let $y \in \tilde{B}\left(x, \dfrac{d}{2}\right)$, that is $\| y - x \| < \dfrac{d}{2}$. Since $\| y - x \| = \sup_{\sigma\in G}\left|y_\sigma - x_\sigma\right|$ and $\| y - x \| < \dfrac{d}{2}$ we have $| y_\sigma - x_\sigma | < \dfrac{d}{2}$, for any $\sigma \in G$.

For $y \in I(a)$, that is $\inf_{\sigma\in G}\left(\left|y_\sigma - a_\sigma\right|\right) > 0$ we obtain

$$\left|y_\sigma - x_\sigma\right| = \left|y_\sigma - a_\sigma - x_\sigma + a_\sigma\right| \ge d.$$

Thus $\inf_{\sigma\in G}\left(\left|y_\sigma - x_\sigma\right|\right) \ge d > 0$, that is the set $I(a)$ is an open set.

Remark 3.7 *Taking into account all we have proved above, the set of points of continuity for* $f: I(a) \to \widetilde{\mathbb{Q}}_p$, $f(x) = \dfrac{1}{x-a}$, *where* $x \neq a$, *is the set* $\left\{ x \in \widetilde{\mathbb{Q}}_p \middle| \inf_{\sigma \in G} \left(|x_\sigma - a_\sigma| \right) > 0 \right\}$.

4. REGULAR POLYNOMIALS IN $\widetilde{\mathbb{Q}}_p$

Let us consider $\widetilde{\mathbb{Q}}_p[X]$. The elements of $\widetilde{\mathbb{Q}}_p[X]$ are polynomials of the type $f(X) = a_0 + a_1 X + \ldots + a_n X^n$, with coefficients in $\widetilde{\mathbb{Q}}_p$.

Consider $f(X) = a_0 + a_1 X + \ldots + a_n X^n$. Then $f_\sigma(X_\sigma) = (a_0)_\sigma + (a_1)_\sigma X_\sigma + \ldots + (a_n)_\sigma X_\sigma^n$ are called the projections of f on $\mathbb{C}_p$, for any $\sigma \in G$.

For $f, g \in \widetilde{\mathbb{Q}}_p[X]$ one obtains $(f \cdot g)_\sigma = f_\sigma \cdot g_\sigma$.

Definition 4.1 $f \in \widetilde{\mathbb{Q}}_p[X]$ *is called regular polynomial if there is a nonzero divisor.*

Proposition 4.2 *The polynomial* $f \in \widetilde{\mathbb{Q}}_p[X]$ *is regular if and only if* $f_\sigma \neq 0$ *for any* $\sigma \in G$.

Proof. Let us consider the regular polynomial $f \in \widetilde{\mathbb{Q}}_p[X]$ defined by $f(X) = a_0 + a_1 X + \ldots + a_n X^n$.

If for $\sigma_0 \in G$, we have $f_{\sigma_0} = 0$, then $f_{\sigma_0}(X_\sigma) = (a_0)_{\sigma_0} + (a_1)_{\sigma_0} X_\sigma + \ldots + (a_n)_{\sigma_0} X_\sigma^n$, we deduce that $(a_0)_{\sigma_0} = (a_1)_{\sigma_0} = \ldots = (a_n)_{\sigma_0} = 0$. Accordingly [6], there exists an element $e \in \widetilde{\mathbb{Q}}_p$ so that $e^2 = e$ and $e \cdot a_i = 0$, for $i = 1, 2, \ldots, n$.

Let $g = e \cdot X$. We have $g \cdot f = (e \cdot X) \cdot f = ea_0 X + ea_1 X^2 + \cdots + ea_n X^{n+1} = 0$, which is a contradiction. Hence $f_\sigma \neq 0$, for any $\sigma \in G$.

Conversely, let $f_\sigma \neq 0$ for any $\sigma \in G$. We suppose that $f \cdot g = 0$, for $f, g \in \widetilde{\mathbb{Q}}_p[X]$. One obtains:

$$f_\sigma \cdot g_\sigma = (f \cdot g)_\sigma, \text{ for any } \sigma \in G.$$

Since $f_\sigma \neq 0$, we have $g_\sigma = 0$, we deduce that $g = 0$. Hence the polynomial $f \in \widetilde{\mathbb{Q}}_p[X]$ is regular.

Corollary 4.3 *Let us consider* $f(X) = a_0 + a_1 X + \ldots + a_n X^n$, *with* $a_i \in \widetilde{\mathbb{Q}}_p$, *for* $i = 0, 1, \ldots, n$. *If exists an element* $a_i \in U(\widetilde{\mathbb{Q}}_p)$, *then* f *is regular.*

Proof. If $f(X) = a_0 + a_1 X + \ldots + a_n X^n$ and $a_i \in U(\widetilde{\mathbb{Q}}_p)$, then $(a_i)_\sigma \neq 0$, for any $\sigma \in G$, we deduce that $f_\sigma(X_\sigma) \neq 0$, hence the polynomial f is regular.

Remark 4.4 *In particulary, the monic polynomial* $f \in \widetilde{\mathbb{Q}}_p[X]$ *is regular.*

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