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MULTIOBJECTIVE FRACTIONAL VARIATIONAL PROBLEMS WITH (ρ, b)-QUASIINVEXITY

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Necessary conditions for normal efficient solutions to a class of multiobjective fractional variational problems (MFP) with nonlinear equality and inequality constraints are established using a parametric approach to relate efficient solutions of a fractional problem and a non-fractional problem. Based on these normal efficiency criteria a Mond-Weir type dual is formulated and appropriate duality theorems are proved assuming (ρ, b)-quasiinvexity of the functions involved.

Key words: Multiobjective fractional variational problem, Efficient solutions,Quasiinvexity, Duality.

1. NOTATION AND STATEMENT OF THE PROBLEM

Let R^n be the *n*-dimensional Euclidean space. Throughout the paper, the following conventions for vectors in R^n will be adopted.

For vectors $v=(v_1,..., v_n)$, $w=(w_1,..., w_n)$ the relations $v=w, v<w,$ $\mathrm{v} \leqq \mathrm{w}$, and $\mathrm{v} \leq \mathrm{w}$ are defined as follows

$$v=w \Leftrightarrow v_i = w_i,\ i=\overline{1,n}\ ;\ v<w \Leftrightarrow v_i < w_i,\ i=\overline{1,n}\ ;$$
$$v \leqq w \Leftrightarrow v_i \leqq w_i,\ i=\overline{1,n}\ \mathbf{;}\ v \leq w \Leftrightarrow u \leqq w \ \textbf{and}\ u \neq v.$$

Let $I=[a,b]$ be a real interval and $f=(f_1,..., f_p): I \times \mathrm{R}^n \times \mathrm{R}^n \to \mathrm{R}^p, k=(k_1,..., k_p)$: $I \times \mathrm{R}^n \times \mathrm{R}^n \to \mathrm{R}^p$, $g=(g_1,...,g_m): I \times \mathrm{R}^n \times \mathrm{R}^n \to \mathrm{R}^m$, $h=(h_1,...,h_q): I \times \mathrm{R}^n \times \mathrm{R}^n \to \mathrm{R}^q$ be twice differentiable functions.

Consider a vector-valued function $f(t, x, \dot{x})$, where $t \in I$ and $x: I \to \mathrm{R}^n$, with derivative $\dot{x}$ with respect to *t*.Denote by f_x and $f_{\dot{x}}$ the $p \times n$ matrices of first-order partial derivatives of f with respect to x and $\dot{x}$, i.e. $f_x = (f_{1x}, f_{2x},..., f_{px})'$ and $f_{\dot{x}} = (f_{1\dot{x}}, f_{2\dot{x}},..., f_{p\dot{x}})'$, with

$$f_{ix} = \left(\frac{\partial f_i}{\partial x_1},..., \frac{\partial f_i}{\partial x_n}\right) \ \textbf{and}\ f_{i\dot{x}} = \left(\frac{\partial f_1}{\partial \dot{x}_1},..., \frac{\partial f_i}{\partial \dot{x}_n}\right), \boldsymbol{i=1,2,\ldots,p}.$$

Similarly, k_x, g_x, h_x and $k_{\dot{x}}, g_{\dot{x}}, h_{\dot{x}}$ denote the $p \times n, m \times n, q \times n$ matrices of the first partial derivatives of *k, g* and *h* respectively, with respect to *x* and $\dot{x}$. Let $C(I,\ \mathrm{R}^n)$ denote the space of piecewise smooth (continuously differentiable) functions *x* with the norm $\|x\| := \|x\|_\infty + \|Dx\|_\infty$, where the differential operator *D* is given by

$$u = Dx \Leftrightarrow x(t) = x(a) + \int_a^t u(s)\mathrm{d}s,$$

where $x(a)$ is a given boundary value. Therefore, $D = \mathrm{d}/\mathrm{d}t$, except at discontinuities.

Consider the multiobjective variational problem

$$\text{(MFP)}\begin{cases}\text{Minimize } \left(\dfrac{\int_a^b f_1(t,x,\dot{x})\,\mathrm{d}t}{\int_a^b k_1(t,x,\dot{x})\,\mathrm{d}t},\dots,\dfrac{\int_a^b f_p(t,x,\dot{x})\,\mathrm{d}t}{\int_a^b k_p(t,x,\dot{x})\,\mathrm{d}t}\right) \\ \text{subject to} \\ \quad x(a) = a_0,\ x(b) = b_0, \\ \quad g(t,x,\dot{x}) \leqq 0,\ h(t,x,\dot{x}) = 0, \forall t \in I\end{cases}.$$

Assume that $\int_a^b k_i(t,x,\dot{x})\,\mathrm{d}t > 0$ for all $i = 1, 2, \cdots, .p.$

Let $\mathbf{D} = \{x \in C(I, \mathrm{R}^n) \mid x(a) = a_0, x(b) = b_0, f(t,x,\dot{x}) \leqq 0, h(t,x,\dot{x}) = 0, \forall t \in I\}$ be the set of all feasible solutions to (MFP).

2. PRELIMINARIES. THE MULTIOBJECTIVE VARIATIONAL PROBLEM

In this section we recall some definitions and auxiliary results that will be needed later in our discussion of efficiency conditions and Mond-Weir duality to (MFP).

Consider the multiobjective variational problem

$$\text{(MP)}\begin{cases}\min \int_a^b f(t,x,\dot{x})\,\mathrm{d}t = \left(\int_a^b f_1(t,x,\dot{x})\,\mathrm{d}t,\dots,\int_a^b f_p(t,x,\dot{x})\,\mathrm{d}t\right) \\ \text{subject to } \quad x(a) = a_0, x(b) = b_0 \\ \qquad\qquad g(t,x,\dot{x}) \leqq 0,\ h(t,x,\dot{x}) = 0,\ t \in I.\end{cases}$$

The domain of (MP) is also **D.**

Definition 2.1. A feasible solution $x^0 \in \mathbf{D}$ is said to be an *efficient solution* to (MP) iff for all feasible solutions $x \in \mathbf{D}$

$$\int_a^b f(t,x,\dot{x})\mathrm{d}t \leqq \int_a^b f(t,x^0,\dot{x}^0)\mathrm{d}t \quad \Rightarrow \quad \int_a^b f(t,x,\dot{x})\mathrm{d}t = \int_a^b f(t,x^0,\dot{x}^0)\mathrm{d}t$$

Let $s : I \times \mathrm{R}^n \times \mathrm{R}^m \to \mathrm{R}$ be a scalar continuously differentiable function and consider now the scalar variational problem

$$\text{(SP)}\begin{cases}\text{Minimize } \int_a^b s(t,x,\dot{x})\,\mathrm{d}t \\ \text{subject to } \ x(a) = a_0, x(b) = b_0 \\ \qquad\qquad g(t,x,\dot{x}) \leq 0, h(t,x,\dot{x}) = 0,\ t \in I.\end{cases}$$

Definition 2.2. The optimal solution $x^0 \in \mathbf{D}$ to (SP) is called *normal* if $\lambda \neq 0$.

According to this definition, without loss of generality, in what follows we can take $\lambda = 1$.

The next result gives necessary Valentine's conditions [4] for the optimality of x^0 to (SP).

Theorem 2.1 (Necessary Valentine's conditions). *Let* x^0 *be a* (*normal*) *optimal solution to* (SP) *and let s, g and h be continuously differentiable functions. Then there exists a scalar* λ *and piecewice smooth functions* $\mu^0(t)$ *and* $\upsilon^0(t)$ *satisfying the conditions*

$$\textbf{(VC)}\begin{cases}\lambda s_x(t,x^0,\dot{x}^0)+\mu^0(t)'g_x(t,x^0,\dot{x}^0)+\upsilon^0(t)'h_x(t,x^0,\dot{x}^0)=\\ =\dfrac{d}{dt}[\lambda s_{\dot{x}}(t,x^0,\dot{x}^0)+\mu^0(t)'g_{\dot{x}}(t,x^0,\dot{x}^0)+\upsilon^0(t)h_{\dot{x}}(t,x^0,\dot{x}^0)]\\ \mu^0(t)'g(t,x^0,\dot{x}^0)=0,\ \ \mu^0(t)\geqq 0,\quad \forall t\in I,\quad (\lambda=1).\end{cases}$$

We have

Lemma 2.2 (Chankong, Haimes [1]). $x^0 \in \mathbf{D}$ *is an efficient solution to problem* (MP) *if and only if* x^0 *is an optimal solution to the scalar problem*

$$\mathbf{P}_i(x^0)\qquad\begin{cases}\text{Minimize } \int_a^b f_i(t,x,\dot{x})\,dt\\ \text{subject to } \ x(a)=a_0,\ x(b)=b_0\\ \qquad\qquad g(t,x,\dot{x})=0,\ h(t,x,\dot{x})=0,\ \ t\in I\\ \int_a^b f_j(t,x,\dot{x})\,dt \leqq \int_a^b f_j(t,x^0,\dot{x}^0)\,dt\ ,\ j=\overline{1,p},\ j\neq i.\end{cases}$$

for each $i=1,\dots,p.$

Lemma 2.3. *If* x^0 *is a* (*normal*) *optimal solution to the scalar problem* $\mathrm{P}_i(x^0)$, *then there exist a scalar* λ_i $(\lambda_i=1)$ *and functions* μ_i *and* υ_i *such that*

$$\begin{cases}\lambda_i f_{ix}(t,x^0,\dot{x}^0)+\mu_i(t)'g_x(t,x^0,\dot{x}^0)+\upsilon_i(t)h_x(t,x^0,\dot{x}^0)=\\ =\dfrac{d}{dt}[\lambda_i f_{i\dot{x}}(t,x^0,\dot{x}^0)+\mu_i(t)'g_{\dot{x}}(t,x^0,\dot{x}^0)+\upsilon_i(t)'h_{\dot{x}}(t,x^0,\dot{x}^0)]\\ \mu_i(t)'g(t,x^0,\dot{x}^0)=0,\ \ \mu_i(t)\geqq 0,\ \ \forall t\in I\\ \lambda_i \geqq 0,\ (\lambda_i=1).\end{cases}\tag{2.1}$$

Theorema 2.4. *Let* $x^0 \in \mathbf{D}$ *be a normal efficient solution to* (MP). *Then there exist a vector* $\lambda^0 \in \mathrm{R}^p$ *and piecewise smooth functions* $\mu^0 : I \to \mathrm{R}^m$ *and* $\upsilon^0 : I \to \mathrm{R}^q$ *that satisfy the Valentine's conditions*

$$\textbf{(MV)}\begin{cases}\lambda^{0\prime} f_x(t,x^0,\dot{x}^0)+\mu^0(t)'g_x(t,x^0,\dot{x}^0)+\upsilon^0(t)h_x(t,x^0,\dot{x}^0)=\\ =\dfrac{d}{dt}\Big[\lambda^{0\prime} f_{\dot{x}}(t,x^0,\dot{x}^0)+\mu^0(t)g_{\dot{x}}(t,x^0,\dot{x}^0)+\upsilon^0(t)h_{\dot{x}}(t,x^0,\dot{x}^0)\Big]\\ \mu^0(t)'g(t,x^0,\dot{x}^0)=0,\ \ \mu_i(t)\geqq 0,\quad \forall t\in I\\ \lambda^0 \geq 0,\ \ e'\lambda^0=1,\ \ e=(1,\dots,1)'\in \mathrm{R}\,.\end{cases}$$

Let $\rho \in \mathrm{R}$ and a function $b : X\times X \to [0,\infty)$. Put

$$H(x)=\int_a^b h(t,x,\dot{x})\,dt$$

Definition 2.3. The function H is said to be (*strictly*) (ρ, b)-*quasiinvex* at x^0 if there exist vector functions $\eta : I \times X \times X \to \mathrm{R}^n$ with $\eta(t, x(t), \dot{x}(t)) = 0$ for $x(t) = x^0(t)$ and $\theta : X \times X \to \mathrm{R}^n$ such that for any $x\ (x \neq x^0)$, $H(x) \leqq H(x^0) \ \Rightarrow$

$$\Rightarrow \mathrm{b}(x, x^0)\int_a^b [\eta' h_x(t, x^0, \dot{x}^0) + (D\eta)' h_{\dot{x}}(t, x^0, \dot{x}^0)]\, dt\ (<) \leqq -\rho b(x, x^0)\left\|\theta(x, x^0)\right\|^2.$$

3. EFFICIENCY NECESSARY CONDITIONS FOR (MFP)

Consider now the problem

$$\textbf{(FP)}_i(x^0)\begin{cases} \min\limits_x \dfrac{\int_a^b f_i(t, x, \dot{x})\,dt}{\int_a^b k_i(t, x, \dot{x})\,dt} \\ \text{subject to} \quad x(a) = a_0,\ x(b) = b_0 \\ \qquad g(t, x, \dot{x}) \leqq 0,\ h(t, x, \dot{x}) = 0,\ t \in I \\ \dfrac{\int_a^b f_j(t, x, \dot{x})\,dt}{\int_a^b k_j(t, x, \dot{x})\,dt} \leqq \dfrac{\int_a^b f_j(t, x^0, \dot{x}^0)\,dt}{\int_a^b k_j(t, x^0, \dot{x}^0)\,dt},\ j = \overline{1, p},\ j \neq i. \end{cases}$$

Denoting

$$R_i^0 = \frac{\int_a^b f_i(t, x^0, \dot{x}^0)\,dt}{\int_a^b k_i(t, x^0, \dot{x}^0)\,dt} = \min_x \frac{\int_a^b f_i(t, x, \dot{x})\,dt}{\int_a^b k_i(t, x, \dot{x})\,dt},\quad i = \overline{1, p},$$

problem $(\mathrm{FP})_i(x^0)$ can be written as

$$\textbf{(FPR)}_i\begin{cases} \min\limits_x \dfrac{\int_a^b f_i(t, x, \dot{x})\,dt}{\int_a^b k_i(t, x, \dot{x})\,dt} \qquad \left[= R_i^0\right] \\ \text{subject to} \quad x(a) = a_0,\ x(b) = b_0 \\ \qquad g(t, x, \dot{x}) \leqq 0,\ h(t, x, \dot{x}) = 0,\quad t \in I \\ \int_a^b [f_j(t, x, \dot{x}) - R_j^0 k_j(t, x, \dot{x})]\,dt \leqq 0,\ j = \overline{1, p},\ j \neq i. \end{cases}$$

Consider now the problem

$$\textbf{(SPR)}_i\begin{cases} \min\limits_{x,u} \int_a^b [f_i(t, x, \dot{x}) - R_i^0 k_i(t, x, \dot{x})]\,dt \\ \text{subject to} \quad x(a) = a_0,\ x(b) = b_0 \\ \qquad g(t, x, \dot{x}) \leqq 0,\ h(t, x, \dot{x}) = 0,\quad t \in I \\ \int_a^b [f_j(t, x, \dot{x}) - R_j^0 k_j(t, x, \dot{x})]\,dt \leqq 0,\ j = \overline{1, p},\ j \neq i. \end{cases}$$

Lemma.3.1 (Jaganathan [2]). $x^0 \in \mathbf{D}$ *is optimal to* $(\mathrm{FPR})_i$ *if and only if* x^0 *is optimal to* $(\mathrm{SPR})_i$.

Theorem 3.2. $x^0 \in \mathbf{D}$ *is an efficient solution for* (MFP) *if and only if it is an optimal solution for each of the problems* (SPR)$_i$, $i=\overline{1,p}$.

Definition 3.1. $x^0 \in \mathbf{D}$ is said to be a *normal efficient solution* of (MP) if it is a normal optimal solution to at least one of the scalar problems (FP)$_i(x^0)$, $i=\overline{1,p}$.

Let a vector $\lambda = (\lambda_1, ..., \lambda_p)' \in R^p$ and functions $\mu : I \to R^q$ and $\upsilon : I \to R^n$. Consider the function $V : I \times R^n \times R^n \times R^m \times R^q \to R$ defined by

$$V(t,x,\lambda,\mu,\upsilon) = \sum_{i=1}^{p} \lambda_i \left[f_i(t,x,\dot{x}) - R_i^0 k_i(t,x,\dot{x}) \right] + \mu(t)' g(t,x,\dot{x}) + \upsilon(t)' h(t,x,\dot{x}).$$

Theorem 3.3 (Necessary efficiency conditions). *Let* $x^0 \in \mathbf{D}$ *be a normal efficient solution to problem* (MFP). *Then there exist* $\lambda^0 \in R^p$ *and piecewise smooth functions* $\mu^0 : I \to R^m$ *and* $\upsilon^0 : I \to R^q$ *that satisfy the conditions*

$$\textbf{(MFV)} \begin{cases} V_x(t,x^0,\lambda^0,\mu^0,\upsilon^0) = \dfrac{d}{dt} V_{\dot{x}}(t,x^0,\lambda^0,\mu^0,\upsilon^0) \\ \mu^0(t) g(t,x^0,\dot{x}^0) = 0,\ \mu^0(t) \geqq 0,\ \forall t \in I \\ \lambda^0 \ge 0, e'\lambda^0 = 1,\ e = (1, ..., 1)' \in R^p. \end{cases}$$

Denote

$$F_i(x^0) = \int_a^b f_i(t,x^0,\dot{x}^0)\, dt, \quad K_i(x^0) = \int_a^b k_i(t,x^0,\dot{x}^0)\, dt.$$

We then have

$$R_i^0 = \frac{F_i(x^0)}{K_i(x^0)}, \quad i = \overline{1,p}$$

Theorem 3.4 (Necessary efficiency conditions). *Let* x^0 *be a normal efficient solution to problem* (MFP). *Then there exist* $\lambda^0 \in R^n$ *and piecewise smooth functions* $\mu^0 : I \to R^q$ *and* $\upsilon^0 : I \to R^n$ *that satisfy the conditions*

$$\text{(MFV)} \begin{cases} \sum_{i=1}^{p} \lambda_i^0 [K_i(x^0) f_{ix}(t,x^0,\dot{x}^0) - F_i(x^0) k_{ix}(t,x^0,x^0)] + \mu^0(t)' g_x(t,x^0,\dot{x}^0) + \upsilon^0(t)' h_x(t,x^0,\dot{x}^0) = \\ = \dfrac{d}{dt} \{ \sum_{i=1}^{p} \lambda_i^0 [K_i(x^0) f_{i\dot{x}}(t,x^0,\dot{x}^0) - F_i(x^0) k_{i\dot{x}}(t,x^0,\dot{x}^0)] + \mu^0(t)' g_{\dot{x}}(t,x^0,\dot{x}^0) + \upsilon^0(t) h_{\dot{x}}(t,x^0,\dot{x}^0) \} \\ \quad \mu^0(t)' g(t,x^0,\dot{x}^0) = 0, \quad \mu^0(t) \geqq 0, \quad \forall t \in I \\ \quad \lambda^0 \ge 0,\ e'\lambda^0 = 1. \end{cases}$$

4. MOND-WEIR TYPE DUALITY

Let $\{J_1, ..., J_r\}$ be a partition of $\{1, ..., m\}$ and $\{K_1, ..., K_r\}$ a partition of $\{1, ..., q\}$. Consider functions $y, v \in C(I, R^n)$. We associate with (MFP) the multiobjective variational problem (MFD)

$$
\text{(MFD)} \begin{cases} \text{Maximize} \left(\dfrac{\int_a^b f_1(t,y,\dot{y})\,dt}{\int_a^b k_1(t,y,\dot{y})\,dt}, \ldots, \dfrac{\int_a^b f_p(t,y,\dot{y})\,dt}{\int_a^b k_p(t,y,\dot{y})\,dt} \right) \\ \text{subject to } \quad y(a)=a_0,\ y(b)=b_0 \\ \sum_{i=1}^{p} \lambda_i [K_i(y) f_{iy}(t,y,\dot{y}) - F_i(y) k_{iy}(t,y,\dot{y})] + \mu(t)' g_y(t,y,\dot{y}) + \upsilon(t)' h_y(t,y,\dot{y}) = \\ = \dfrac{d}{dt} \{ \sum_{i=1}^{p} \lambda_i [K_i(y) f_{i\dot{y}}(t,y,\dot{y}) - F_i(y) k_{i\dot{y}}(t,y,y)] + \mu(t)' g_{\dot{y}}(t,y,\dot{y}) + \upsilon(t)' h_{\dot{y}}(t,y,\dot{y}) \} \\ \mu_{J_\alpha}(t)' g_{J_\alpha}(t,y,\dot{y}) + \upsilon_{K_\alpha}(t) h_{K_\alpha}(t,y,\dot{y}) \geqq 0, \quad \alpha = \overline{1,r}, \quad \forall t \in I \\ \lambda \geq 0,\ e'\lambda = 1. \end{cases}
$$

Denote by π (MFP) $= \pi(x)$ the value of problem (MFP) at $x \in \mathbf{D}$ and let δ(MFD) $= \delta(y,\lambda,\eta,\upsilon)$ be the value of the dual (MFD) at $(y,\lambda,\eta,\upsilon) \in \Delta$, where Δ is the domain of (MFD).

Theorem 4.1 (Weak duality). *Let* x *and* (y,λ,μ,v) *be feasible points of problems* (MFP) *and* (MFD). *Assume that are satisfied the conditions* :

a) *for each* $i=\overline{1,p}$ *we have* $F_i(x)>0,\ K_i(x)>0.,\ \forall x \in X.$

b) *for each* $i=\overline{1,p}$, $F_i(x)$ *is* (ρ_i',b)*-quasiinvex at* y *and* $-K_i(x)$ *is* (ρ_i'',b)*-quasiinvex at* y*, all with respect to* η *and* θ.

c) $\int_a^b [\mu_{J_\alpha}(t)' g_{J_\alpha}(t,x,u) + \upsilon_{K_\alpha}(t)' h_{K_\alpha}(t,x,\dot{x})]\,dt$ *is* (ρ_α''',b)*-quasiinvex at* y *with respect to* η *and* θ.

d) *one of the functions of* b)-c) *is strictly* (ρ,b)*-quasiinvex*

e) $\sum_{i=1}^{p} \lambda_i [\rho_i' K_i(y) + \rho_i'' F_i(y)] + \sum_{\alpha=1}^{r} \rho_\alpha''' \geqq 0$.

Then $\pi(x) \leq \delta(y,\lambda,\mu,v)$ *is false.*

Theorem 4.2 (Direct duality). *Let* x^0 *be a normal efficient solution to the primal* (MFP) *and assume the hypotheses of Theorem* 4.1. *Then there exist* $\lambda^0 \in \mathrm{R}^p$ *and piecewise smooth functions* $\mu^0 : I \to \mathrm{R}^m$ *and* $\upsilon^0 : I \to \mathrm{R}^r$ *such that* $(x^0,\lambda^0,\mu^o,\upsilon^0)$ *is an efficient solution to the dual problem* (MFD) *and, moreover,* $\pi(x^0) = \delta(x^0,\lambda^0,\mu^o,\upsilon^0)$.

Theorem 4.3 (Converse duality). *Let* $(x^0,\lambda^0,\mu^0,\upsilon^0)$ *be an efficient solution to the dual problem* (MFD) *and assume the conditions*:

a^0) $\bar{x}$ *is a normal efficient solution to the primal* (MFP).

b^0) *for each* $i=\overline{1,p}$, $F_i(x^0)>0,\ K_i(x^0)>0.$

c^0) *for each* $i=\overline{1,p}$, $F_i(x)$ *is* (ρ_i',b)*-quasiinxex at* x^0*, and* $-K_i(x)$ *is* (ρ_i'',b)*-quasiinvex at* x^0*, all with respect to* η *and* θ.

$d^0)$ $\int_a^b [\mu_{J_\alpha}(t)' g_{J_\alpha}(t,x,\dot{x}) + \upsilon_{K_\alpha}(t)' h_{K_\alpha}(t,x,\dot{x})]\,\mathrm{d}t$ *is* (ρ'''_α, b)*-quasiinvex at* (x^0, u^0) *with respect to* η *and* θ.

$e^0)$ *one of the functions of* $b^0) - c^0)$ *is strictly* (ρ, b)*-quasiinvex.*

$f^0)$ $\sum_{i=1}^{p} \lambda_i^0 [\rho'_i K_i(x^0) + \rho''_i F_i(x^0)] + \sum_{\alpha=1}^{r} \rho'''_\alpha \geqq 0.$

Then $\bar{x} = x^0$ *and, moreover,* $\pi(x^0) = \delta(x^0, \lambda^0, \mu^0, \upsilon^0)$.

The proofs will appear in [3].

REFERENCES

1. CHANKONG, V., HAIMES,Y.Y., *Multiobjective decisions making: theory and methodology,* North Holland, New York, 1983.
2. JAGANNATHAN, R.,*Duality for nonlinear fractional programming* ,Z. Oper.Res.,**17** (1973) 1-3.
3. MITITELU, S., STANCU-MINASIAN, I.M., *Efficiency and duality for multiobjective fractional variational problems with* (ρ, b) *-quasiinvexity*, (submitted).
4. VALENTINE, F. A., *The problem of Lagrange with differentiable inequality as added side condition,*in Contributions to the Calculus of Variations, 1933-37 Univ. of Chicago Press, 1937, 407- 448.

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