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NOTES ON THE ISOMORPHIC MODULAR GROUP ALGEBRAS OF P-SPLITTING AND P-MIXED ABELIAN GROUPS

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We give new proofs of two isomorphism theorems for commutative modular group algebras published in (Comm. Alg., 2000), (Compt. rend. Acad. bulg. Sci., 2002) and (Le Matem., 2006) by usage of a crucial isomorphism relation established in (Proc. Roman. Acad. Sci., Ser. A – Math., 2006).

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Throughout the present paper, let KG be the group algebra of a multiplicative abelian group G with p-primary component G_p over a field K of characteristic $p \neq 0$. Denote by $S(KG)$ the p-component of torsion of the group $V(KG)$ of all normalized (i.e. of augmentation 1) units. In [3] and, more generally, in [4] we proved the following.

Theorem A. *If G is p-splitting, that is, G_p is a direct factor of G, G_p is simply presented and K is perfect, then $S(KG)/G_p$ is simply presented. Moreover, $S(KG)$ is simply presented with a direct factor G_p. In particular, if G is in addition p-mixed, that is, the only torsion is p-torsion, G is a direct factor of $V(KG)$ with simply presented complement*.
With the aid of this statement, the following isomorphism assertion was also proved there.

Theorem B. *If G is p-splitting whose G_p is simply presented and $KH \cong KG$ as K-algebras for some other group H, then $H_p \cong G_p$. In particular, if in addition G is p-mixed, $H \cong G$.*
Likewise, in [2] we established the following affirmation.

Theorem C. *Suppose that G is a coproduct of countable abelian groups and K is perfect. Then $S(KG)/G_p$ is a coproduct of countable abelian groups and G_p is a direct factor of $S(KG)$. Thus $S(KG)$ is a coproduct of countable abelian groups. Moreover, if $KH \cong KG$ as K-algebras for some group H, then $H_p \cong G_p$. In particular, if G is in addition p-mixed, G is a direct factor of $V(KG)$ with the same complementary factor. Thus $V(KG)$ is a coproduct of countable abelian groups. Moreover, if $KH \cong KG$ as K-algebras for some group H, there exists a coproduct of countable abelian p-groups T such that $H\times T \cong G\times T$. Thus H is a coproduct of p-mixed countable abelian groups.*
With this at hand, we obtained in [5] the following isomorphism claim.

Theorem D. *Let G be a p-mixed coproduct with finite torsion-free rank of countable abelian groups and $KH \cong KG$ as K-algebras for another group H. Then $H \cong G$.*

Our aim in this article is to give an independent smooth confirmation of both Theorems B and D by using the following simple but significant technicality of [6].

Proposition E. *$KG \cong KH$ as K-algebras implies that $K(S(KG)/G_p) \cong K(S(KH)/H_p)$ as K-algebras.*

And so, we are now prepared to give the desired proofs, which are, indeed, more direct than the existing ones and give another strategy in the advantage of greater clarity.

Proof of Theorem B. Without loss of generality we may assume that $KG = KH$ because $KG \cong KH$ yields that $KG = KH_1$ for some $H_1 \cong H$. According to Theorem A we have that $S(KG)/G_p$ is simply presented, hence Proposition E combined with [9] lead us to $S(KH)/H_p$ is simply presented. Since H_p is balanced in S(KH), it is its direct factor with simply presented complement. But by Theorem A we have $S(KG) = S(KH)$ is simply presented, whence so is H_p as a direct factor. Furthermore, [8] applies to get that the maximal divisible subgroups of G_p and H_p are respectively isomorphic as well as the Ulm-Kaplansky invariants of G_p and H_p are respectively equal. Consequently, G_p and H_p are isomorphic.

Next, if G is p-mixed, $KG = KH$ obviously implies that H is p-mixed. Besides, $G = G_p \times M$ for some subgroup M secures that $V(FG) = S(FG) \times V(FM)$ and hence $V(FH) = S(FH) \times V(FM)$. Since by what we have shown above H_p is a direct factor of $S(KH)$, it is readily seen that H_p is a direct factor of V(KH), thus of H (cf. [1] and [4] too). Finally, $G \cong G_p \times (G/G_p)$ and $H \cong H_p \times (H/H_p)$. Employing [8], we deduce that $G/G_p \cong H/H_p$. That is why, $G \cong H$ and we are done.

As usual, we denote by $I(KG; G_p)$ the relative augmentation ideal of KG with respect to G_p.

If the isomorphism implication $G_p \cong H_p \Rightarrow (1+I(KG; G_p))/G_p \cong (1+I(KG; H_p))/H_p$ holds true, another idea for a proof without Proposition E may be like this: Owing to Theorem A, we find that $S(KG)=S(KH)$ is simply presented. So, we apply [7] to infer that H_p is simply presented. Henceforth, we can copy the idea from the preceding proof to get $G_p \cong H_p$. Furthermore, observing that $S(KG)= 1+I(KG; G_p)$ and $S(KH)= 1+I(KH; H_p)$, where $KG = KH$, by what we have assumed above we derive that $S(KG)/G_p = (1+I(KG; G_p))/G_p \cong (1+I(KG; H_p))/H_p = S(KH)/H_p$ and thus, in virtue of Theorem A, $S(KH)/H_p$ is simply presented. Further, the proof goes on as in the foregoing proof.

Proof of Theorem D. Invoking to Theorem C, $S(KG)/G_p$ is a direct sum of countable groups. But Proposition E along with [9] allows us to conclude that $S(KH)/H_p$ is also a direct sum of countable groups. The balanced property of H_p in $S(KH)$ insures that H_p is a direct factor of $S(KH)$. But, because H is p-mixed as is G, it is immediate that $V(KH)=HS(KH)$ and thus H is a direct factor of $V(KH)$. Again Theorem C enables us that both $V(KH)$ and H are direct sums of countable groups. Hereafter, the proof goes on in the same way as in [5]. □

With the aid of Theorem C the same idea for proof of Theorem D, without Proposition E, may also be demonstrated.

Remark. Results (a) and (c) from [10] imitated in a weaker form our theorems in [3] and [4] (compare with the statements presented above) as well as results from [7].

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