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L-MOMENTS EVALUATION FOR IDENTICALLY AND NONIDENTICALLY WEIBULL DISTRIBUTED RANDOM VARIABLES

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In this paper we compute the L-moments for identically and nonidentically Weibull distributed random variables. The results are based on the evaluation of moments of order statistics. First, we derive the order statistics and L-moments for generalized Weibull distributed random variables. As a special case, we obtain the expression for the order statistics and L-moments for the Weibull distribution. Moreover the L-moments for identically distributed Weibull random variabes could be obtained from the L-moments for nonidentically Weibull distributed random variables.

Key words: L-moments; order statistics; Weibull distributed random variables.

1. INTRODUCTION

It is a standard practice in statistics to use descriptive measures (quantities) in order to describe the shape of the distribution of a population. Such measures has been defined by means of classical moments of different orders: the mean to estimate location, the variance to measure the spread, the standardized measures for skewness and kurtosis.

Though it is well-known that despite their popularity both in data description and more formal statistical procedures, sample moments suffer from several drawbacks. For example, they are sensitive to extreme observations. Moreover, the asymptotic variances of moment-based estimators are mainly determined by higher order moments, which are rather large or even unbounded for heavy tail distributions. As a consequence, asymptotic efficiency of sample moments is rather poor especially for distributions with fat tails.

Moment-based measures are just particular, but not exhaustive means of summarizing qualitative features of the shape of a distribution. The notions of dispersion, skewness and kurtosis are rather abstract and therefore can be described in many ways.

An alternative set of very effective descriptive measures called L-moments are based on partial orderings and seem to largely overcome the sampling drawbacks of classical measures.

The L-moments appeared for the first time – without name – in quantile expansion of Sillitto [10] and then in Hosking's [5] research report. Formally, the L-moments were introduced by Hosking [4] as a linear combination of the order statistics of a population.

Like classical moments, L-moments provide intuitive information about the shape of a distribution, which can be consistently estimated from their sample values. Hosking, Wallis and Wood [7] and Hosking and Wallis [6] use L-moments estimation method to extreme value distribution. They found that it performs better than method of moments and, moreover, that both moment- based methods do well in small samples compared to maximum likelihood estimation.

Taking into account the more satisfactory sampling behavior of L-moments estimators, the superior empirical performance over some data sets and the results of formal simulation experiments over some selected distributions, some authors (for example, Hosking [3]) recommend the use of L-moments instead of the classical moments.

In what follows we will remind the formal definition of L-moments and some important results concerning them. In Sections 3 we derive the L-moments for generalized Weibull distributed random variables. As a particular case, in Section 4 we find the L-moments expression for identically Weibull distributed random variables. In Section 5 we evaluate the L-moments for nonidentically Weibull distributed random variables. Also we note that the results from Section 4 could be obtained as a special case of those from Section 5.

2. SOME PRELIMINARY RESULTS

Let X be a real-valued random variable with cumulative distribution function $F(x)$ and let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the order statistics of a random sample of size n from the distribution of X.

Definition 2.1 ([3])**.** The L-moments are defined to be the quantities

$$\lambda_r = \frac{1}{r}\sum_{j=1}^{r-1}(-1)^j C_{r-1}^j E\left(X_{r-j:r}\right), \tag{2.1}$$

for $r = 1,2,\ldots,n$, where $C_n^r = \frac{n!}{r!(n-r)!}$, $n \in \mathrm{N}^*$.

Remark 2.1. The L-moments as well as ordinary moments are special cases of probability weighted moments introduced by Greenwood et al. [2] as

$$M_{p,r,s} = E\left(X^p (F(X))^r (1-F(X))^s\right), \tag{2.2}$$

where $p,r,s \ge 0$. Obviously $M_{p,0,0}$ are the ordinary moments, $M_{1,r,0} = \frac{1}{r+1}E\left(X_{r+1:r+1}\right)$ and

$$M_{p,j-1,r-j} = \frac{(j-1)!(r-j)!}{r!}E\left(X_{j:r}^p\right).$$

The use of L-moments to describe probability distributions is justified by the following result.

Theorem 2.1 (Hosking [3])**.**

(i) The L-moments λ_r, $r = 1,2,..,n$, of a real-valued random variable X exist if and only if X has finite mean.

(ii) A distribution whose mean exists is characterized by its L-moments $\left(\lambda_r\right)_{r=1,2,\ldots}$.

Thus a distribution can be specified by its L-moments even if some of its conventional moments do not exist. Furthermore, such a specification is always unique, which is of course not true for conventional moments.

The following result will be used in Section 4.

Lemma 2.1 (Khan et al. [8])**.** If X is a positive random variable with cumulative distribution function $F(x)$ and $J_k(a,b) = \int_0^\infty x^{k-1}F(x)^b(1-F(x))^a dx$, where a,b are real numbers chosen such that $J_k(a,b)$ exists and is finite, then

$$J_k(n-r+1, r-1) = \sum_{i=1}^{r}(-1)^{r-i} C_{r-1}^{i-1} J_k(n-i+1, 0).$$

In Section 5 we will use the following result.

Theorem 2.2 (Barakat and Abdelkader [1])**.** Let $X_1, X_2, ..., X_n$ be nonidentically distributed random variables and $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ the corresponding order statistics. Then for $r \ge 1$ and $k = 1,2,...$ the *k*th order moment of $X_{r:n}$ can be evaluated recursively as

$$E\left(X_{r:n}^{(k)}\right) = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} C_{j-1}^{n-r} I_j^{(k)} ,$$

where $I_j^{(k)} = \sum\sum_{1 \le j_1 < j_2 < ... < j_i \le n} ... \sum \alpha_{1:i(j)}^{(k)}$, $\alpha_{1:i(m)}^{(k)} = k \int_0^\infty x^{k-1} \prod_{t=1}^{m} \left(1 - F_{i_t}(x)\right) dx$ is the *k*th moment of the minimum of $\left(X_{i_1}, X_{i_2}, ..., X_{i_m}\right)$ and $F_i(x)$ is the cumulative distribution function of X_i.

3. GENERALIZED WEIBULL DISTRIBUTION CASE

Let consider a random sample $X_1, X_2, ..., X_n$ of size n from a generalized Weibull distribution with cumulative distribution function

$$F(x) = \left(1 - \exp\left(-\frac{x^p}{\theta}\right)\right)^{\frac{1}{\gamma}}, \tag{3.1}$$

where $x > 0$, $p, \theta, \gamma \in \mathrm{R}_+^*$. The probability density function in this case is

$$f(x) = \frac{p}{\theta\gamma} x^{p-1} \left(1 - (F(x))^{\gamma}\right)(F(x))^{1-\gamma} . \tag{3.2}$$

The probability density function of the *r*th order statistics $X_{r:n}$ is

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} \frac{p}{\theta\gamma} x^{p-1} \left[F(x)^{r-\gamma} - F(x)^r\right](1 - F(x))^{n-r} . \tag{3.3}$$

Denote

$$\alpha_{r:n}^{(s)} = E\left(X_{r:n}^s\right). \tag{3.4}$$

Taking into account relation (3.3), the expression of $J_k(a,b)$ from Lemma 2.1, the fact that

$$\alpha_{r:n}^{(k-p)} = E\left(X_{r:n}^{k-p}\right) = \int_0^\infty x^{k-p} f_{r:n}(x) dx$$

and Definition 2.1, we get the following result for the *k*th order moment of the *r*th order statistic and the L-moments for the generalized Weibull distribution.

Theorem 3.1. Under the previously mentioned conditions,

$$\text{i) } \alpha_{r:n}^{(k-p)} = \frac{n!}{(r-1)!(n-r)!} \frac{p}{\theta\gamma} \left[J_k(n-r, r-\gamma) - J_k(n-r, r)\right] \tag{3.5}$$

$$\text{ii) } \lambda_r = \frac{p}{\theta\gamma} \sum_{j=0}^{r-1} (-1)^j \left(C_{r-1}^j\right)^2 \left[J_{p+1}(j, r-j-\gamma) - J_{p+1}(j, r-j)\right], \tag{3.6}$$

for $r = 1,2,..., n$, if the quantities involved in the above expression exist and are finite.

4. THE WEIBULL DISTRIBUTION CASE

In this section we consider a random sample $X_1, X_2, ..., X_n$ of size n from a two-parameter Weibull distribution with cumulative distribution function

$$F(x) = 1 - \exp\left(-\frac{x^p}{\theta}\right), \tag{4.1}$$

where $x > 0$ and $p, \theta \in \mathrm{R}_+^*$. Obviously, this corresponds to the case $\gamma = 1$ from Section 3. Therefore, the probability density function in this case is

$$f(x) = \frac{p}{\theta} x^{p-1}(1 - F(x)). \tag{4.2}$$

Applying Theorem 3.1.i) for $\gamma = 1$ we get

$$\alpha_{r:n}^{(k-p)} = \frac{n!}{(r-1)!(n-r)!}\frac{p}{\theta}\left[J_k(n-r, r-1) - J_k(n-r, r)\right].$$

It is easy to see from relation the expression of $J_k(a,b)$ given in Lemma 2.1 that

$$J_k(a,b) = J_k(a, b-1) - J_k(a+1, b-1) \tag{4.3}$$

and, therefore,

$$\alpha_{r:n}^{(k-p)} = \frac{n!}{(r-1)!(n-r)!}\frac{p}{\theta} J_k(n-r+1, r-1), \tag{4.4}$$

a result also obtained directly by Khan et al. [8].

Moreover, in this particular case we have the following results regarding L-moments.

Theorem 4.1. The L-moments for the Weibull distribution satisfy

$$\text{i) } \lambda_r = \frac{p}{\theta}\sum_{j=0}^{r-1}(-1)^j \left(C_{r-1}^j\right)^2 J_{p+1}(j+1, r-j-1), \tag{4.5}$$

and

$$\text{ii) } \lambda_r = \frac{\theta^{\frac{1}{p}}}{p}\Gamma\left(\frac{1}{p}\right)\sum_{i=1}^{r}(-1)^{r-i}\left(\frac{1}{r-i+1}\right)^{\frac{p+1}{p}} a(i,r), \tag{4.6}$$

where $a(i,r) = \sum_{j=0}^{r-i}\left(C_{r-1}^j\right)^2 C_{r-j-1}^{i-1}$, for $r = 1,2,...,n$.

Proof. i) We use Definition 2.1 and relation (4.4) or Theorem 3.1.ii) and relation (4.3).

ii) By Lemma 1 from Khan et al. [8], we get

$$J_{p+1}(j+1, r-j-1) = \sum_{i=1}^{r-j}(-1)^{r-j-i} C_{r-j-1}^{i-1} J_{p+1}(r-i+1, 0). \tag{4.7}$$

Furthermore, we note that

$$J_k(a,0) = \left(\frac{\theta}{a}\right)^{\frac{k}{p}}\frac{1}{p}\Gamma\left(\frac{k}{p}\right). \tag{4.8}$$

Now, using Theorem 4.1.i), and relations (4.7) and (4.8), we get relation (4.6).

The next result gives a simple expression for the L-moments in a special case.

Proposition 4.1. For $\theta = 1$ and $p = 1$, the L-moments for the Weibull distribution are given by

$$\lambda_r = \frac{1}{r(r-1)}, \tag{4.9}$$

where $r = 2,...,n$.

Proof. Sarhan [9] showed that for $\theta = 1$ and $p = 1$ we have

$$\alpha_{r:n} = \sum_{i=1}^{r} \frac{1}{n-i+1}.$$

By the definition of L-moments, we then have

$$\lambda_r = \frac{1}{r}\sum_{j=0}^{r-1}(-1)^j C_{r-1}^j \sum_{i=1}^{r-j}\frac{1}{r-i+1} = \frac{1}{r}\sum_{i=1}^{r}\frac{1}{r-i+1}\sum_{j=0}^{r-i}(-1)^j C_{r-1}^j = \frac{1}{r(r-1)}\sum_{i=2}^{r}(-1)^{r-i} C_{r-1}^{i-2}$$

from which we get relation (4.9) for $r = 2,...,n$.

Remark 4.1. Obviously, $\lambda_1 = E(X) = \alpha_{1:1} = \theta$ for $p = 1$.

5. L-MOMENTS FOR NONIDENTICALLY WEIBULL DISTRIBUTED RANDOM VARIABLES

Let $X_1, X_2,...,X_n$ be independent positive random variables with cumulative distribution functions $F_1, F_2,...,F_n$, where

$$F_i(x) = 1 - \exp\left(-\frac{x^p}{\theta_i}\right), \tag{5.1}$$

with $x > 0$, $p, \theta_1, \theta_2,...,\theta_n \in \mathrm{R}_+^*$.

As in the previous sections, let $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ denote the corresponding order statistics, $\alpha_{r:n} = E(X_{r:n})$, and λ_r the L-moments defined by (2.1).

The next result gives a characterization of L-moments for nonidentically Weibull distributed random variables.

Theorem 5.1. The L-moments for nonidentically distributed Weibull random variables are given by

$$\lambda_r = \frac{1}{p}\Gamma\left(\frac{1}{p}\right)\sum_{i=1}^{r}(-1)^{i-1} b(i,r)\tilde{I}_i, \tag{5.2}$$

where $b(i,r) = \sum_{j=0}^{i-1}\frac{1}{r}C_{r-1}^j C_{i-1}^j$ and $\tilde{I}_i = \sum\sum ... \sum_{1 \le j_1 < j_2 < ... < j_i \le r}\left(\frac{1}{\theta_{j_1}^{-1} + \theta_{j_2}^{-1} + ... + \theta_{j_i}^{-1}}\right)^{\frac{1}{p}}$.

Proof. We apply Theorem 2.2 of Barakat and Abdelkader [1] for $k = 1$. Thus, we have to evaluate $I_j^{(1)} \overset{not}{=} I_j$. Taking into account the expression for $I_j^{(k)}$ and relation (5.1), we obtain $I_j = \frac{1}{p}\Gamma\left(\frac{1}{p}\right)\tilde{I}_j$, where $\tilde{I}_j$ has the form mentioned above. Now, we get

$$\alpha_{r-j:r} = \frac{1}{p}\Gamma\left(\frac{1}{p}\right)\sum_{i=j+1}^{r}(-1)^{i-j-1}C_{i-1}^{j}\tilde{I}_i \ ,$$

and then, from Definition 2.1, after some calculation,

$$\lambda_r = \frac{1}{p}\Gamma\left(\frac{1}{p}\right)\sum_{i=1}^{r}(-1)^{i-1}\left[\frac{1}{r}\sum_{j=0}^{i-1}C_{r-1}^{j}C_{i-1}^{j}\right]\tilde{I}_i \ ,$$

hence, finally, relation (5.2).

Remark 5.1. For identically Weibull distributed random variables, that is, $\theta_i = \theta$ for $\forall i \in \overline{1,r}$, from Theorem 5.1, we get relation (4.6) from Theorem 4.1.

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