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THE ANALYSIS OF A SYSTEM OF RIGID BODIES'S DYNAMICS BY LINEAR EQUIVALENCE METHOD

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The problem of a system of rigid bodies subjected to holonomous, bilateral frictionless constraints and punctual contacts with dry friction between some bodies of the system is analysed in this paper. The Jean and Pratt problem of a 2D system of bodies is solved by using the linear equivalence method (LEM).

Key words: multibody system, holonomous constraints, dry friction, LEM.

1. INTRODUCTION

Many publications deal with analysis of the mechanical system of rigid bodies with dry friction between some bodies of the system. For example, in [1] the dynamics of mechanical systems with dry friction elements modeled by set-valued force laws are described by differential inclusions. An equilibrium set of such a differential inclusion corresponds to a stationary mode for which the friction elements are sticking. Popa and coworkers [2] analyze the motion of multibody hybrid systems characterized by switching between constraints, which are defined as different dynamical regimes. Jean and Pratt [3] and Moreau [4], [5] have developed the general motion equations for a system of rigid bodies submitted to usual constraints and forces, and to punctual contacts with dry friction between some bodies of the system. In this paper the nonlinear equation of the Jean and Pratt problem is solved by using linear equaivalence method (LEM). The LEM was introduced by Toma in studying both qualitatively and quantitatively the nonlinear dynamical systems and their solutions (Toma [6]-[8]). LEM and its applications are extensively presented by Toma in a monograph (Toma [9]).

2. EQUATIONS OF MOTION

Consider a system composed by a finite set of rigid bodies of generalized coordinates $q \in \mathbb{R}^n$. The initial conditions are given

$$q(t_0) = q_0,\ \dot{q}(t_0) = v_0,\ q_0, v_0 \in \mathbb{R}^n,\ t_0 \in \mathbb{R}, \tag{2.1}$$

with $\| q - q_0 \| \le \varepsilon$, $\| v - v_0 \| \le \varepsilon$, $\varepsilon > 0$ a very small number. In the spirit of the Jean and Pratt [3], we say that the motion is defined on the time interval $t \in [t_0, t_0 + T]$, $0 < T \le T_0$, $t_0, T_0 \in \mathbb{R}$, where T must be find. The contact points are denoted by P_j, $j = 1, 2, ..., m$. The unknowns of the problem are T, the mapping $t \to q(t)$, $t \in [t_0, t_0 + T]$, $0 < T \le T_0$, $t_0, T_0 \in \mathbb{R}$, which describes the motion of the system, the mapping $t \to f(q(t)) \in S$, $S = S_1 \times ... \times S_m$, $S_j = \mathbb{R}^2$, $t \in [t_0, t_0 + T]$, $0 < T \le T_0$, $t_0, T_0 \in \mathbb{R}$ which describes the pair $f_j(q(t))$ of the components of the tangential reaction force at a contact point P_j characterized by dry friction, and the mapping $t \to N(q(t)) \in \mathbb{R}^m$, $t \in [t_0, t_0 + T]$, $0 < T \le T_0$, $t_0, T_0 \in \mathbb{R}$ which describes components $N_j(q(t))$ of the normal reaction force. We say that (2.1) is a non-critical initial condition, if there exist strictly positive real numbers r_1, r_2, $0 < r_1 \le r_2$, such that $\forall t, q, \dot{q}$, $\forall i = 1,2,...,m$, $r_1 \le M_i(t,q,\dot{q}) \le r_2$, where $M_i(t,q,\dot{q})$ represent the normal components of the reaction when there is perfect slip.

By definition $M(t,q(t),\dot{q}(t))$ is the mapping $(t,q,\dot{q}) \to M(t,q,\dot{q}) \in \mathbb{R}^m$. Certainly, M_i is strictly positive and bounded in a neighbourhood of t_0, q_0 and v_0. In the neighbourhood of a non-critical initial condition, unilateral constraints are maintained when there is perfect slip, and $M(t_0, q_0, v_0) > 0$.

If (2.1) is a non-critical initial condition, the usual frictionless holonomous, bilateral constraints is satisfied if there exists $0 < k < 1$ such that

$$-k/(1-k)r_1 + r_2 > 0, \ \bar{L}\bar{d}R \le k/m, \tag{2.2}$$

where $\bar{d}R$ is an upper bound of the friction coefficients. The kinetic and potential energies of the system are given by

$$T = 1/2\dot{q}(t)A(t,q(t))\dot{q}(t), \ V = 1/2q(t)B(t,q(t))q(t), \tag{2.3}$$

where $A(t,q(t))$ is the mapping $(t,q) \to A(t,q) \in L(\mathbb{R}^n, \mathbb{R}^n)$ representing the matrix of kinetic energy, L is the Hilbert space of the 2-power integrable functions, and $B(t,q(t))$ is the mapping $(t,q) \to B(t,q) \in L(\mathbb{R}^n, \mathbb{R}^n)$ representing the matrix of potential energy. The Lagrangian has the form $L = T - V$, and the general form of the Lagrange equations of the system subject to holonomic frictionless constraints depending or not the time, can be written as

$$A(t,q(t))\ddot{q}(t) = R(t,q(t))f(q(t)) + F(t,q(t)),$$

where $R(t,q(t))$ is the mapping $(t,q) \to R(t,q) \in L(S, \mathbb{R}^n)$, and $R(t,q(t))f(t)$ are the generalized forces representing the reaction force, and $F(t,q(t))$ is the mapping $(t,q) \to F(t,q) \in \mathbb{R}^n$.

The components of the sliding velocities are given by $U(t) = R(t,q(t))\dot{q}(t) + \tilde{U}(t,q(t))$, with $\tilde{U}(t,q(t))$ the mapping $(t,q) \to \tilde{U}(t,q) \in S$. The components $N_j(t)$ of the normal reaction forces are expressed as $N(t) = \tilde{L}(t,q(t))f(q(t)) + M(t,q(t),\dot{q}(t))$, where $\tilde{L}(t,q(t))$ is the mapping $(t,q) \to \tilde{L}(t,q) \in L(S, \mathbb{R}^m)$. We add the condition of a unilateral constraint to be maintained at the contact points P_j, $j = 1,2,...,m$, with dry friction between some bodies of the system $N(t) \in \Gamma$, where Γ is the interior of the positive cone of $\mathbb{R}^m$, i.e. $\Gamma = \{n \in \mathbb{R}^m, \forall i = 1,2,...,m, \ n_i > 0\}$.

The formulation of the friction law Coulomb's law is

$$-f(q) \in \partial\phi(q(t), N(t), U(t)), \tag{2.4}$$

where $\partial\phi$ is the subdifferential with respect to w of the mapping $(t,q,n,\dot{w})$, $n \in \Gamma$, $w \in S \to \phi(t,q,n,w) \in \mathbb{R}$ convex with respective to $\dot{w}$.

As an example, let us consider the Jean and Pratt problem [3]. The mass m moves with respect to mass M along the line C of slope $\tan\alpha$, of the mass M with frictionless bilateral constraints. The mass m also is sliding on the horizontal line B of a fixed body. Friction at the point of contact P is obeying Coulomb's law. The mass m is subjected to a horizontal constant force with magnitude $-Mg\tan\alpha$. The mass M moves vertical along the vertical line A with a frictionless bilateral constraint (fig.2.1). The generalized coordinate $q(t)$ is the abscissa of P. We denote with $f(q(t))$ the horizontal component of the reaction force. The motion equation (2.3) becomes

$$A\ddot{q}(t) = f(q(t)), \ A = m + M\tan^2\alpha, \tag{2.5}$$

subjected to initial condition (2.1). The sliding velocity $U(t)$ and the normal component of an unilateral constraint $N(q(t))$ of the reaction force from the line B upon the mass m are

$$U(t)=\dot{q}(t)\,,\ N(t)=-M\tan\alpha\ddot{q}(t)+Mg+mg\,,\ N(t)>0\,.$$

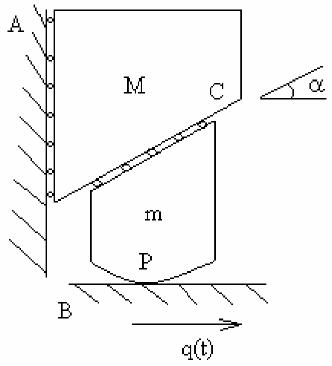

Fig. 2.1. The Jean and Pratt system.

We suppose that the Coulomb's law, can be written in the virtue of (2.4), as

$$f(q)\in[-RN(t),RN(t)]\,,\ (f_1-f)U(t)\geq 0\,,\ \forall f_1\in[-RN(t),RN(t)]\,, \tag{2.6}$$

where R is the friction coefficient, and

$$\left(f/N_0\right)_{,q}+\left(f/N_0\right)^2=\text{const.}\,,\ N_0=\frac{Ag(m+M)}{A\mp RM\tan\alpha}. \tag{2.7}$$

The $\mp$ refers to the initial data $v_0>0$ and respectively, $v_0<0$. We have $M(t,q(t),\dot{q}(t))=Mg+mg$ and $\tilde{L}(t,q)=M\tan\alpha/A$. Consequently, any initial condition (2.1) is non-critical. The condition (2.2) becomes $\dfrac{RM\tan\alpha}{A}<1$. This condition is verified for $v_0>0$, and may be or not for $v_0<0$. By using the notation $f/(RN_0)=\tanh Bq$, $B=RN_0/A$, the conditions (2.6) and (2.7) are automatically verified for $\text{const}=1$, and the motion equation (2.5) becomes

$$\ddot{q}(t)=B\tanh(Bq(t))\,,\ B=RN_0/A\,. \tag{2.8}$$

3. SOLUTIONS

Solutions of (2.8) and initial conditions (2.1) are determined by using the linear equivalence method developed by Toma. To understand the method, let us consider the system of ordinary differential equations

$$\dot{y}=f(t,y),\ f(t,y)\equiv\left[f_j(t,y)\right]_{j=\overline{1,n}},\ y\in\left(\mathrm{C}^1(\mathrm{I})\right)^n,\ \mathrm{I}=[a,b]\subseteq\mathbb{R}\,, \tag{3.1}$$

where $f_j(t,y)$ are analytic functions with respect to y, uniformly on I

$$f_j(t,y)=\sum_{|\mu|=1}^{\infty}f_{j\mu}(t)\,y^{\mu},\ j=1,\ldots,n,\ \mu\in\left(\mathbb{N}\cup\{0\}\right)^n,$$

and their coefficients $f_{j\mu}:\mathrm{I}\to\mathbb{R}$ are at least of class $\mathrm{C}^0(\mathrm{I})$. Even if the equations are not homogeneous, we shall reduce it to homogeneous equations of the form (3.1). The system (3.1) may be also written as

$$\dot{y}-f(t,y)=0, \tag{3.2}$$

LEM considers an exponential mapping depending on n parameters $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$

$$v(x,\xi) \equiv \exp\langle \xi, y \rangle, \tag{3.3}$$

that associates to initial equations, two linear equivalent equations: (i) a linear partial differential equation of first order with respect to x

$$Lv(x,\xi) \equiv \dot{v} - \langle \xi, f(x,D) \rangle v = 0, \tag{3.4}$$

and (ii) a linear first order differential equation

$$\dot{v}_\gamma = \sum_{j=1}^{n} \gamma_j \sum_{|\mu|=1}^{m_j} f_{j\mu}(t) v_{\gamma+\mu-e_j}, \quad e_j = \left(\delta_i^j\right)_{i=\overline{1,n}}. \tag{3.5}$$

The equation (3.4) was obtained by differentiating (3.3) with respect to t and replacing the derivatives $\dot{y}_j$ from the nonlinear system. The usual notation $f_j(t, \mathrm{D}_\xi)$ stands for the formal operator

$$f_j(t, \mathrm{D}_\xi) = \sum_{|\mu|=1}^{\infty} f_{j\mu}(t) \frac{\partial^{|\mu|}}{\partial \xi^\mu}.$$

The formal scalar product in (3.4) is expressed as

$$\sum_{j=1}^{n} \xi_j f_j(t, D_\xi) \equiv \langle \xi, f(t,D) \rangle.$$

The second equation (3.5) is obtained from the first one, by searching the unknown function v in the class of analytic in ξ functions

$$v(t,\xi) = 1 + \sum_{|\gamma|=1}^{\infty} v_\gamma(t) \frac{\xi^\gamma}{\gamma!}.$$

Consider now for (3.1), the initial conditions

$$y(t_0) = y_0, \; t_0 \in \mathrm{I}, \tag{3.6}$$

which can be written by applying $v(t_0,\xi) = \exp\langle \xi, y_0 \rangle, \; \xi \in \mathbb{R}^n$, as $v_\gamma(t_0) = y_0^\gamma, \; |\gamma| \in \mathbb{N}$.

In order to get back to the solutions of the initial nonlinear Cauchy problem, the partial differential equation (3.5) can be defined on some space of analytic with respect to ξ functions, uniformly for $t \in \mathrm{I}$. Now, let us return to the Jean and Pratt problem. The nonlinear equation (2.8)

$$\ddot{q} = B \tanh Bq, \tag{3.7}$$

must be solved under conditions

$$q(t_0) = 0, \; \dot{q}(t_0) = v_0. \tag{3.8}$$

The problem (3.7) is similar to Troesch's plasma problem of the confinement of plasma by radiation pressure (Toma [10]), with the difference that the governing equation in the Troesch's problem is $y_{,xx} = \tanh y$. Using the same technique as the Troesch's problem solving, we introduce changes of function and variable $x = Bt, \; y(x) = Bq$, and the equation (3.7) and (3.8) become

$$y_{,xx} = \tanh y, \; y(0) = 0, \; y_{,x}(0) = z_0. \tag{3.9}$$

Writing (3.9) in the form of a first order differential equations

$$y_{,x} = z, \; z_{,x} = \tanh y, \tag{3.10}$$

and applying the LEM exponential mapping

$$v(x,\sigma,\xi)=\exp(\sigma y+\xi z), \tag{3.11}$$

the first LEM equivalent equation is obtained

$$v_{,x}=\sigma v_{,\xi}+\xi\tanh(D_\sigma)v, \tag{3.12}$$

where

$$\tanh(D_\sigma)\equiv\sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}D_\sigma^k v,\ D_\sigma^k\equiv\frac{\partial^k v}{\partial\sigma^k}, \tag{3.13}$$

with B_{2k} the Bernoulli numbers (Abramovitch and Stegun [11]). The initial conditions (3.9) become

$$y(0)=0,\ z(0)=z_0. \tag{3.14}$$

By considering for v the expansion in the form

$$v(x,\sigma,\xi)=1+\sum_{j+k=1}^{\infty}v_{jk}(x)\frac{\sigma^j}{j!}\frac{\xi^k}{k!}, \tag{3.15}$$

equation (3.12) leads to the second LEM equivalent equations

$$v'_{jk}=jv_{j-1,k}+k\sum_{m=1}^{\infty}\frac{2^m(2^{2m-2}-1)B_{2m-2}}{(m-1)!}v_{2m-1-k,k}. \tag{3.16}$$

This linear infinite system may be written in matrix form

$$\mathbf{V}_{,x}=\mathbf{AV},\ \mathbf{V}=(\mathbf{V}_{2m-1})_{m\in\mathbb{N}},\ \mathbf{V}_{2m-1}=(v_{jk}(x))_{j+k=2m-1}, \tag{3.17}$$

where the matrix $\mathbf{A}$ has the form

$$\mathbf{A}=\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{13} & \mathbf{A}_{15} & \dots\\ \mathbf{0} & \mathbf{A}_{33} & \mathbf{A}_{35} & \dots\\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{55} & \dots\\ \dots & \dots & \dots & \dots\end{bmatrix}, \tag{3.18}$$

with the square diagonal cells $\mathbf{A}_{2m-1,2m-1}$. The initial conditions (3.14) become for $\mathbf{V}$

$$\mathbf{V}(0,z_0)=(\mathbf{V}_{2j-1}(0,z_0))_{j\in\mathbb{N}}. \tag{3.19}$$

To solve (3.17) and (3.19), we truncate the system, by introducing the projectors P_m, such as $P_m\mathbf{V}=\mathbf{V}^{(m)}$, $\mathbf{V}^{(m)}=(\mathbf{V}_{2j-1})_{j=1,\dots,m}$. Therefore, the finite truncated systems may be written in the form $\frac{\mathrm{d}\mathbf{V}^{(m)}}{\mathrm{d}x}=\mathbf{A}^{(m)}\mathbf{V}^{(m)}$, $\mathbf{A}^{(m)}$ being truncated matrices, up to order m inclusive. The initial conditions for $\mathbf{V}^{(m)}$ are $\mathbf{V}^{(m)}(0,z_0)=(\mathbf{V}_{2j-1}(0,z_0))_{j=1,\dots,m}$. After elementary algebra, we obtain for (3.9) the approximating solution

$$y(x)\cong z_0x-\frac{1}{2}x^2+\frac{4}{5z_0}x^3-\frac{3}{5z_0^2}x^4+\frac{32}{75z_0^3}x^5\dots. \tag{3.20}$$

4. CONCLUSIONS

For m, R and α fixed, if the initial conditions are $q(t_0) = q_0$, $\dot{q}(t_0) = v_0$, $v_0 > 0$, the single solution of the problem is given by (3.20)

$$q(t) \cong v_0 t - \frac{B}{2}t^2 + \frac{4}{5}\left(\frac{B^2}{v_0}\right)t^3 - \frac{3}{5}\left(\frac{B^3}{v_0^2}\right)t^4 + \frac{32}{75}\left(\frac{B^4}{v_0^3}\right)t^5 - \ldots, \; B = \frac{RN_0}{m + M\tan^2\alpha}, \tag{4.1}$$

for $t \in [0, \tau_0]$, $\tau_0 = \frac{v_0}{B} = \frac{v_0(m + M\tan^2\alpha)}{RN_0}$ and $N_0 = \frac{g(m+M)(m + M\tan^2\alpha)}{m + M\tan^2\alpha - RM\tan\alpha}$. For $t = \tau_0$ we have $q = 0.42\frac{v_0^2}{B}$.

If $t > \tau_0$, $v_0 = 0$, $q(t) = 0$, $f(t) = 0$, $N(t) = 0$. For $R > \tan\alpha$, $\frac{RM\tan\alpha}{m + M\tan^2\alpha} \to 1$, $N_0 \to \infty$ and $\tau_0 \to 0$. It results from here that in the limit process the shocks may occur, as suggested in [3]. The solution (4.1) exists only if the condition (2.2) is satisfied. Indeed, for $v_0 > 0$ this condition is satisfied because f is negative, N is positive, and $f = -RN$ which implies $\frac{RM\tan\alpha}{m + M\tan^2\alpha} < 1$. If $v_0 < 0$, it results $N_0 = \frac{g(m+M)(m + M\tan^2\alpha)}{m + M\tan^2\alpha + RM\tan\alpha}$, and (2.2) may be satisfied or not. The single solution of the problem in this case is

$$q(t) \cong v_0 t + \frac{B}{2}t^2 + \frac{4}{5}\left(\frac{B^2}{v_0}\right)t^3 - \frac{3}{5}\left(\frac{B^3}{v_0^2}\right)t^4 + \frac{32}{75}\left(\frac{B^4}{v_0^3}\right)t^5 - \ldots, \tag{4.2}$$

for $t \in [0, \tau_0]$. For $t > \tau_0$, it follows $q(t) = 0$, $f(t) = 0$ and $N(t) = mg + Mg$.

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