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AN EXISTENCE RESULT OF WEAK SOLUTIONS TO DIRICHLET PROBLEM FOR NONLINEAR SECOND ORDER SYSTEMS OF DIVERGENCE TYPE

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We obtain an existence result for the weak solutions in the Sobolev space $\boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$, $p>2$, to the Dirichlet problem for a nonlinear second order system of divergence type. In fact, it is proved that, in certain hypotheses, the operator naturally associated to the Dirichlet problem is a bounded and coercive Gårding operator [10]. We get a generalization of the results obtained in [4] for the Dirichlet problem of nonlinear elastostatics.

Key words: Sobolev spaces; weak solutions; Gårding operator.

4. SOME FEW PRELIMINARIES

A. The summation over repeated subscripts is understood and the notation $i=\overline{p,q}$, where $p\le q$ are integers, means that the index i takes the values $p, p+1,\dots,q$.

If $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^k$ then $\boldsymbol{a}\cdot\boldsymbol{b}$ is the standard scalar product on $\mathbb{R}^k$ and $|\boldsymbol{a}|$ is the corresponding Euclidean norm of $\boldsymbol{a}$. $\mathbb{M}_{m\times n}$ denotes the linear space of matrices $A=(a_{ij})$ of elements $a_{ij}\in\mathbb{R}$, $i=\overline{1,m}$, $j=\overline{1,n}$. The application $(A,B)\mapsto \mathrm{tr}(AB^T)$, $A,B\in\mathbb{M}_{m\times n}$, is the standard inner product on $\mathbb{M}_{m\times n}$ and $|A|$ is the corresponding norm of A.

B. Throughout this paper we suppose $\Omega\subset\mathbb{R}^n$ is a bounded Lipschitz domain ([1], [3], [5], [9]) with boundary $\partial\Omega$, and $\mathrm{d}\boldsymbol{x}$ denotes the Lebesgue measure on Ω.

We use the notation [6] $\boldsymbol{L}^p(\Omega,\mathbb{R}^m)$, $\boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$, and $\boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m)$, $p\in[1,\infty)$, for the Banach spaces of $\mathbb{R}^m$-valued functions $u=(u_1,\dots,u_m):\Omega\to\mathbb{R}^m$, with components $u_k:\Omega\to\mathbb{R}$, $k=\overline{1,m}$, belonging to Banach spaces $L^p(\Omega)$, $W^{1,p}(\Omega)$, and $W_0^{1,p}(\Omega)$ respectively. $\boldsymbol{L}^p(\Omega,\mathbb{R}^m)$ is a Banach space, separable for $p\in[1,\infty)$ and reflexive for $p\in(1,\infty)$, with respect to the norm

$$\boldsymbol{u}\mapsto\|\boldsymbol{u}\|_{0,p}\equiv\|\boldsymbol{u}\|_p:=\Big(\int_\Omega|\boldsymbol{u}|^p\,\mathrm{d}\boldsymbol{x}\Big)^{1/p}\in[0,\infty),\quad \boldsymbol{u}\in\boldsymbol{L}^p(\Omega,\mathbb{R}^m).$$

If $(\boldsymbol{u},\boldsymbol{v})\in\boldsymbol{L}^p(\Omega,\mathbb{R}^m)\times\boldsymbol{L}^{p'}(\Omega,\mathbb{R}^m)$, $p\in(1,\infty)$, $1/p+1/p'=1$, then the function $(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{x}):=\boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{x})$, $\boldsymbol{x}\in\Omega$, belongs to $L^1(\Omega)$ [7] and it holds the Hölder inequality

$$\int_\Omega\boldsymbol{u}\cdot\boldsymbol{v}\mathrm{d}\boldsymbol{x}\le\int_\Omega|\boldsymbol{u}||\boldsymbol{v}|\,\mathrm{d}\boldsymbol{x}\le\|\boldsymbol{u}\|_p\|\boldsymbol{u}\|_{p'}.$$

The dual of $\boldsymbol{L}^p(\Omega,\mathbb{R}^m)$ is $\boldsymbol{L}^{p'}(\Omega,\mathbb{R}^m)$, i.e. $(\boldsymbol{L}^p(\Omega,\mathbb{R}^m))'=\boldsymbol{L}^{p'}(\Omega,\mathbb{R}^m)$, and duality pairing on $\boldsymbol{L}^p(\Omega,\mathbb{R}^m)\times\boldsymbol{L}^{p'}(\Omega,\mathbb{R}^m)$ is defined by

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\mathrm{d}\boldsymbol{x}.$$

The Sobolev space $\boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$ ([1]–[3], [5], [6], [9]) is separable for $p\in[1,\infty)$ and reflexive for $p\in(1,\infty)$, with respect to the norm

$$\boldsymbol{u}\mapsto\|\boldsymbol{u}\|_{1,p}:=\{\int_\Omega[|\boldsymbol{u}|^p+|\nabla\boldsymbol{u}|^p]dx\}^{1/p}=(\|\boldsymbol{u}\|_p^p+\|\nabla\boldsymbol{u}\|_p^p)^{1/p}\in[0,\infty).$$

Here $\nabla\boldsymbol{u}$ is the *distributional gradient* of $\boldsymbol{u}$, i.e.

$$\nabla\boldsymbol{u}=(\nabla\boldsymbol{u})_{ij}:\Omega\to\mathbb{M}_{m\times n},\quad (\nabla\boldsymbol{u})_{ij}:=D_ju_i,$$

D_ju_i is the j-th partial generalized derivative of u_i. $\boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m)$ is a closed subspace of $\boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$ and, in view of Poincare's inequality ([2], [3], [6]), $\|\boldsymbol{u}\|_p\le k\|\nabla\boldsymbol{u}\|_p$, $\boldsymbol{u}\in\boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m)$, where

$$\boldsymbol{u}\mapsto\|\nabla\boldsymbol{u}\|_p:=(\int_\Omega|\nabla\boldsymbol{u}|^p\,\mathrm{d}\boldsymbol{x})^{1/p}\in[0,\infty),\quad \boldsymbol{u}\in\boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m),$$

is a norm equivalent with the norm $\|\cdot\|_{1,p}$ on $\boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m)$.

In our hypothesis on Ω we have the completely continuous imbedding ([2], [9])

$$\boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)\subset\boldsymbol{L}^p(\Omega,\mathbb{R}^m),\qquad p\in(1,\infty),\tag{1.1}$$

and for $p>2$ the following continuous and dense imbeddings

$$\boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m)\subset\boldsymbol{L}^p(\Omega,\mathbb{R}^m)\subset\boldsymbol{L}^2(\Omega,\mathbb{R}^m)\subset\boldsymbol{W}^{-1,p'}(\Omega,\mathbb{R}^m),\tag{1.2}$$

where $\boldsymbol{W}^{-1,p'}(\Omega,\mathbb{R}^m):=(\boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m))'$. If $p>2$ then $p'\in(1,2)$ and therefore

$$\boldsymbol{L}^p(\Omega,\mathbb{R}^m)\subset\boldsymbol{X}(\Omega,\mathbb{R}^m):=\boldsymbol{L}^{p'}(\Omega,\mathbb{R}^m)\cap\boldsymbol{W}^{-1,p}(\Omega,\mathbb{R}^m)\subset\boldsymbol{W}^{-1,p'}(\Omega,\mathbb{R}^m).\tag{1.3}$$

The weak convergence in $\boldsymbol{L}^p(\Omega,\mathbb{R}^m)$, denoted by $\boldsymbol{u}_n\rightharpoonup\boldsymbol{u}$ in $\boldsymbol{L}^p(\Omega,\mathbb{R}^m)$, is defined by $\int_\Omega\boldsymbol{u}_n\cdot\boldsymbol{u}\mathrm{d}\boldsymbol{x}\to\int_\Omega\boldsymbol{u}\cdot\boldsymbol{v}\mathrm{d}\boldsymbol{x}$, $\forall\boldsymbol{v}\in\boldsymbol{L}^{p'}(\Omega,\mathbb{R}^m)$, while the weak convergence in $\boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$, denoted by $\boldsymbol{u}_n\rightharpoonup\boldsymbol{u}$ in $\boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$, is equivalent with ([5], [6])

$$\boldsymbol{u}_n\rightharpoonup\boldsymbol{u}\text{ and }D_i\boldsymbol{u}_n\rightharpoonup D_i\boldsymbol{u},i=\overline{1,m},\text{ in }\boldsymbol{L}^p(\Omega,\mathbb{R}^m),$$

and implies the strong convergence $\boldsymbol{u}_n\to\boldsymbol{u}$ in $\boldsymbol{L}^p(\Omega,\mathbb{R}^m)$ (Rellich Theorem [5]).

The quotient space $\boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)/\boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m)$ is isomorphic to $\boldsymbol{W}^{1/p',p}(\partial\Omega,\mathbb{R}^m)$ in the sense of the *trace operator* ([2], [3], [9]).

C. The *divergence operator* on the set of mappings $\boldsymbol{S}=(S_{ij}):\Omega\to\mathbb{M}_{m\times n}$, with $S_{ij}\in W^{1,p}(\Omega)$ is defined by

$$\boldsymbol{S}\mapsto\operatorname{div}\boldsymbol{S}:\Omega\to\mathbb{R}^m,\quad(\operatorname{div}\boldsymbol{S})_i:=D_iS_{ij}\subset L^p(\Omega).$$

D. Definition 1.1 *Let $V=(V,\|\cdot\|)$ and $U=(U,\|\cdot\|_U)$, $V\subset U$, be two separable and reflexive Banach spaces. Suppose that V is dense in U and that the imbeding $V\subset U$ is completely continuous* [1]. *The operator $\Lambda:V\to V'$, where V' is the topological dual of V, is said to be a* Gårding operator [10] *if $\Lambda(v)=F(v,v)$, $\forall v\in V$, where the operator $F(\cdot,\cdot):V\times V\to V'$ satisfies the conditions*:

(i) *For every* $w \in V$, $F(\cdot, w): V \to V'$ *is hemicontinuous* [8], *i.e. the real function* $t \mapsto \langle v, F(u+tv, w)\rangle \in \mathbb{R}$, $t \in \mathbb{R}$, *is continuous for every* $u, v, w \in V$, $\langle \cdot, \cdot \rangle$ *being the pairing duality on* $V \times V'$.

(ii) *There exists a continuous function* $\gamma : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$, *satisfying the condition* $\lim_{\theta \downarrow 0} [\theta^{-1} \gamma(x, \theta y)] = 0$, $\forall x, y \in \mathbb{R}^+$, *such that* $\langle u - v, \Lambda(u) - F(v, u)\rangle \geq -\gamma(r, \| u - v \|_U)$, *for every* $u, v \in B_\gamma(0) = \{w \in V : \| w \| < r\}$.

(iii) *If* $u_n \rightharpoonup u$ in V, *the conditions*

$$\begin{cases} \liminf_{n \to \infty} \langle u_n - u, F(v, u_n) - F(v, u)\rangle \geq 0, \\ \liminf_{n \to \infty} \langle w, F(v, u_n) - F(v, u)\rangle \geq 0, \qquad \forall u, v, w \in V. \end{cases}$$

hold simoultaneously.

One shows [10] that a bounded Gårding operator is a *pseudomonotone operator* [8].

Definition 1.2 *An operator* $\Lambda : V \to V'$ *is said to be* coercive [8] *if*

$$\| v \|^{-1} < v, \Lambda(v) > \to \infty \text{ as } \| v \| \to \infty. \tag{1.4}$$

THEOREM 1.1 ([10]) *If* V *is a reflexive and separable Banach space and* $\Lambda : V \to V'$ *is a bounded and coercive Gårding operator then* Λ *is surjective, i.e. for every* $f \in V'$ *the operator equation* $\Lambda(u) = f$ *has at least a solution* $u \in V$.

2. SECOND ORDER SYSTEMS OF DIVERGENCE TYPE

We consider the following second order system of divergence type [9]

$$-\operatorname{div} \boldsymbol{S}(\boldsymbol{u}, \nabla \boldsymbol{u}) - \boldsymbol{b}(\boldsymbol{u}, \nabla \boldsymbol{u}) = \boldsymbol{f} \tag{2.1}$$

in the unknown function $\boldsymbol{u} = (u_1, \ldots, u_m) : \Omega \to \mathbb{R}^m$ from $\boldsymbol{W}^{1,p}(\Omega, \mathbb{R}^m)$, $p > 2$, where $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain and $\boldsymbol{f} = (f_1, \ldots, f_m) : \Omega \to \mathbb{R}^m$,

$$\boldsymbol{x} = (x_1, \ldots, x_n) \mapsto \boldsymbol{S}(\boldsymbol{u}, \nabla \boldsymbol{u})(\boldsymbol{x}) := \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla \boldsymbol{u}(\boldsymbol{x})) \in \mathbb{M}_{m \times n}, \boldsymbol{x} \in \Omega \subset \mathbf{R}^{\mathbf{n}}, \tag{2.2}$$

$$\boldsymbol{x} = (x_1, \ldots, x_{\mathbf{n}}) \mapsto \boldsymbol{b}(\boldsymbol{u}, \nabla \boldsymbol{u}(\boldsymbol{x})) := \boldsymbol{b}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla \boldsymbol{u}(\boldsymbol{x})) \in \mathbb{R}^m, \boldsymbol{x} \in \Omega \subset \mathbb{R}^n, \tag{2.3}$$

are given functions.

Now we present the restrictions imposed to mappings (2.2) and (2.3) for the *solvability* of the system (2.1) in $\boldsymbol{W}^{1,p}(\Omega, \mathbb{R}^m)$, $p > 2$.

(I) Restrictions on $\boldsymbol{S}(\cdot, \cdot)$**.** $a)$ For every $(\boldsymbol{p}, \boldsymbol{P}) \in \mathbb{R}^m \times \mathbb{M}_{m \times n}$, the mapping $\boldsymbol{S}(\cdot, \boldsymbol{p}, \boldsymbol{P}) : \Omega \to \mathbb{M}_{m \times n}$ is (Lebesgue) measurable, i.e. its real components $S_{ij}(\cdot, \boldsymbol{p}, \boldsymbol{P}) : \Omega \to \mathbb{R}$, $i = \overline{1, m}$, $j = \overline{1, n}$ are measurable. $b)$ For almost every (a.e.) $\boldsymbol{x} \in \Omega$ the mapping $\boldsymbol{S}(\boldsymbol{x}, \cdot, \cdot) : \mathbb{R}^m \times \mathbb{M}_{m \times n} \to \mathbb{M}_{m \times n}$ is Fréchet continuously differentiable. This implies that for a.e. $\boldsymbol{x} \in \Omega$ there exist the "*partial derivatives*" of $\boldsymbol{S}$ with respect to $\boldsymbol{p} \in \mathbb{R}^m$ and $\boldsymbol{P} \in \mathbb{M}_{m \times n}$, i.e. the linear operator

$$\begin{cases} (\boldsymbol{p},\boldsymbol{P}) \mapsto \dfrac{\partial \boldsymbol{S}}{\partial \boldsymbol{p}}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P}) \in L(\mathbb{R}^m,\mathbb{M}_{m\times n}), \\ (\boldsymbol{p},\boldsymbol{P}) \mapsto \dfrac{\partial \boldsymbol{S}}{\partial \boldsymbol{P}}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P}) \in L(\mathbb{R}^m,\mathbb{M}_{m\times n}) \end{cases} \tag{2.4}$$

which are continuous on $\mathbb{R}^m \times \mathbb{M}_{m\times n}$ and are defined by

$$\begin{cases} \boldsymbol{q} \;=\; (q_i) \mapsto \dfrac{\partial \boldsymbol{S}}{\partial \boldsymbol{p}}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P})\boldsymbol{q} := \dfrac{\partial \boldsymbol{S}}{\partial p_i}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P})q_i \in \mathbb{M}_{m\times n}, \quad \boldsymbol{q} \in \mathbb{R}^m, \\ \boldsymbol{Q} \;=\; (Q_{ij}) \mapsto \dfrac{\partial \boldsymbol{S}}{\partial \boldsymbol{P}}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P})\boldsymbol{Q} := \dfrac{\partial \boldsymbol{S}}{\partial P_{ij}}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P})Q_{ij} \in \mathbb{M}_{m\times n}, \quad \boldsymbol{Q} \in \mathbb{M}_{m\times n}. \end{cases} \tag{2.5}$$

In (2.4) $L(U,V)$ denotes the space of linear operators from the linear space U to the linear space V. c) For every $(\boldsymbol{p},\boldsymbol{P}) \in \mathbb{R}^m \times \mathbb{M}_{m\times n}$ the mappings

$$\frac{\partial \boldsymbol{S}}{\partial p_i}(\cdot,\boldsymbol{p},\boldsymbol{P}), \frac{\partial \boldsymbol{S}}{\partial P_{ij}}(\cdot,\boldsymbol{p},\boldsymbol{P}) : \Omega \to \mathbb{M}_{m\times n}, \qquad i=\overline{1,m},\, j=\overline{1,n},$$

are measurable. d) Suppose that for every $(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P}) \in \Omega \times \mathbb{R}^m \times \mathbb{M}_{m\times n}$ and $i=\overline{1,m}$, $j=\overline{1,n}$, the following *growth conditions* hold:

$$\begin{cases} |\boldsymbol{S}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P})| \;\le\; \varphi(\boldsymbol{x}) + a^1|\boldsymbol{p}| + a^2|\boldsymbol{P}|, \\ |\dfrac{\partial \boldsymbol{S}}{\partial p_i}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P})| \;\le\; \varphi_i(\boldsymbol{x}) + a_i^1|\boldsymbol{p}| + a_i^2|\boldsymbol{P}|, \\ |\dfrac{\partial \boldsymbol{S}}{\partial P_{ij}}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P})| \;\le\; \varphi_{ij}(\boldsymbol{x}) + a_{ij}^1|\boldsymbol{p}| + a_{ij}^2|\boldsymbol{P}|, \end{cases} \tag{2.6}$$

where the real functions φ, φ_i, φ_{ij} are from $L^p(\Omega)$ and a^1, a^2; a_i^1, a_i^2; a_{ij}^1, a_{ij}^2 are positive constants independent of $(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P})$.

Remark 2.1 *We notice that the conditions* $(\mathbf{I})_a$ *and* $(\mathbf{I})_b$ (*it is required only the continuity of* $\boldsymbol{S}(\boldsymbol{x},\cdot,\cdot)$ *for a.e.* $\boldsymbol{x} \in \Omega$) *shows that* (2.2) *satisfies the* Caratheodory conditions ([9], [11]). *If moreover the condition* $(2.6)_1$ *holds then the* (Nemytsky) *operator* $\boldsymbol{u} \mapsto \boldsymbol{S}(\boldsymbol{u},\nabla\boldsymbol{u})$ *is a well defined* bounded continuous *operator from* $\boldsymbol{L}^p(\Omega,\mathbb{R}^m)$ *into* $\boldsymbol{L}^p(\Omega,\mathbb{R}^m)$ [11]; *in particular this operator is* bounded and continuous *from* $\boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$ *into* $\boldsymbol{L}^p(\Omega,\mathbb{R}^m)$.

Remark 2.2 *If the mapping* (2.2) *satisfies all the conditions* ($\mathbf{I}$), *then*

$$\boldsymbol{u} \mapsto -\operatorname{div} \boldsymbol{S}(\boldsymbol{u},\nabla\boldsymbol{u}) \tag{2.7}$$

is a well defined continuous *operator from* $\boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$ *into* $\boldsymbol{W}^{-1,p}(\Omega,\mathbb{R}^m)$ (*see* [3], [12]), *and taking into account the* Green's formula *in Sobolev spaces* [3] *we obtain*

$$\left\langle \boldsymbol{v}, -\operatorname{div} \boldsymbol{S}(\boldsymbol{u},\nabla\boldsymbol{u}) \right\rangle = \int_\Omega \boldsymbol{S}(\boldsymbol{u},\nabla\boldsymbol{u}) \cdot \nabla \boldsymbol{v} d\boldsymbol{x}, \quad \boldsymbol{v} \in \boldsymbol{W}_0^{1,p'}(\Omega,\mathbb{R}^m), \tag{2.8}$$

where $\langle \cdot,\cdot \rangle$ *is the pairing duality of* $\boldsymbol{W}^{1,p'}(\Omega,\mathbb{R}^m)$ *and* $\boldsymbol{W}^{-1,p}(\Omega,\mathbb{R}^m)$.

We note that if $p>2$ then $\boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m) \subset \boldsymbol{W}_0^{1,p'}(\Omega,\mathbb{R}^m)$ and therefore we have

$$\langle \boldsymbol{v}, -\operatorname{div} \boldsymbol{S}(\boldsymbol{u},\nabla \boldsymbol{u})\rangle = \int_{\Omega} \boldsymbol{S}(\boldsymbol{u},\nabla \boldsymbol{u}) \cdot \nabla \boldsymbol{v} \mathrm{d}\boldsymbol{x}, \tag{2.8'}$$

for every $(\boldsymbol{u},\boldsymbol{v}) \in \boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m) \times \boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$.

Consequently, if restrictions (**I**) hold an $p > 2$, it results that the operator (2.7) determines in a unique way the bounded and continuous operator [11]

$$\begin{cases} \boldsymbol{u} \mapsto A(\boldsymbol{u}) \in \boldsymbol{W}^{-1,p}(\Omega,\mathbb{R}^m), \quad \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m), \\ \langle \boldsymbol{v}, A(\boldsymbol{u})\rangle = \int_{\Omega} \boldsymbol{S}(\boldsymbol{u},\nabla \boldsymbol{u}) \cdot \nabla \boldsymbol{v} \mathrm{d}\boldsymbol{x}, \ \boldsymbol{v} \in \boldsymbol{W}_0^{1,p'}(\Omega,\mathbb{R}^m). \end{cases} \tag{2.9}$$

(II) Restrictions on $\boldsymbol{b}(\boldsymbol{u},\nabla \boldsymbol{u})$. *a*) For every $(\boldsymbol{p},\boldsymbol{P}) \in \mathbb{R}^m \times \mathbb{M}_{m\times n}$ the mapping $\boldsymbol{b}(\cdot,\boldsymbol{p},\boldsymbol{P}) : \Omega \to \mathbb{R}^m$ is measurable, i.e. its real components $b_i(\cdot,\boldsymbol{p},\boldsymbol{P})$ are measurable. *b*) For a.e. $\boldsymbol{x} \in \Omega$, the mapping

$$\boldsymbol{b}(\boldsymbol{x},\cdot,\cdot) : \mathbb{R}^m \times \mathbb{M}_{m\times n} \to \mathbb{R}^m$$

is Fréchet continuously differentiable. This implies that for a.e. $\boldsymbol{x} \in \Omega$ there exist the "*partial derivatives*" of $\boldsymbol{b}$ with respect to $\boldsymbol{p} \in \mathbb{R}^m$ and $\boldsymbol{P} \in \mathbb{M}_{m\times n}$

$$\begin{cases} (\boldsymbol{p},\boldsymbol{P}) \mapsto \dfrac{\partial \boldsymbol{b}}{\partial p_i}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P}) \in \boldsymbol{L}(\mathbb{R}^m \times \mathbb{R}^m), \\ (\boldsymbol{p},\boldsymbol{P}) \mapsto \dfrac{\partial \boldsymbol{b}}{\partial \boldsymbol{P}}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P}) \in \boldsymbol{L}(\mathbb{M}_{m\times n} \times \mathbb{R}^m), \end{cases} \tag{2.10}$$

which are continuous on $\mathbb{R}^m \times \mathbb{M}_{m\times n}$ and are defined through

$$\begin{cases} \boldsymbol{q} = (q_i) \mapsto \dfrac{\partial \boldsymbol{b}}{\partial \boldsymbol{p}}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P})\boldsymbol{q} := \dfrac{\partial \boldsymbol{b}}{\partial p_i}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P}) q_i \in \mathbb{R}^m, \qquad \boldsymbol{q} \in \mathbb{R}^m, \\ \boldsymbol{Q} = (Q_{ij}) \mapsto \dfrac{\partial \boldsymbol{b}}{\partial \boldsymbol{P}}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P})\boldsymbol{Q} := \dfrac{\partial \boldsymbol{b}}{\partial P_{ij}}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P}) Q_{ij} \in \mathbb{R}^m, \ \boldsymbol{Q} \in \mathbb{M}_{m\times n}. \end{cases} \tag{2.11}$$

c) For each $(\boldsymbol{p},\boldsymbol{P}) \in \mathbb{R}^m \times \mathbb{M}_{m\times n}$ the mappings

$$\frac{\partial \boldsymbol{b}}{\partial p_i}(\cdot,\boldsymbol{p},\boldsymbol{P}) : \Omega \to \mathbb{R}^m, \frac{\partial \boldsymbol{b}}{\partial P_{ij}}(\cdot,\boldsymbol{p},\boldsymbol{P}) : \Omega \to \mathbb{M}_{m\times n}, \quad i = \overline{1,m},\ j = \overline{1,n},$$

are measurable. *d*) The mapping $\boldsymbol{b}(\cdot,\cdot,\cdot)$ satisfies the growth condition

$$|\boldsymbol{b}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P})| \le \psi(\boldsymbol{x}) + b^1 |\boldsymbol{p}|^{p-1} + b^2 |\boldsymbol{P}|^{p-1}, \qquad \forall (\boldsymbol{x},\boldsymbol{p},\boldsymbol{P}) \in \Omega \times \mathbb{R}^m \times \mathbb{M}_{m\times n}, \tag{2.12}$$

where $\psi \in \boldsymbol{L}^p(\Omega)$ and $b^1 > 0$, $b^2 > 0$ are constants independent of $(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P})$.

Remark 2.3 *The condition* $(\mathbf{II})_a$ *and the continuity of* $\boldsymbol{b}(\boldsymbol{x},\cdot,\cdot)$ *for a.e.* $\boldsymbol{x} \in \Omega$ *shows that the mapping* (2.3) *satisfies the Caratheodory conditions. If moreover the growth condition* (2.12) *holds it results that the* (Nemytsky) *operator*

$$\boldsymbol{u} \mapsto B(\boldsymbol{u}) := \boldsymbol{b}(\boldsymbol{u},\nabla \boldsymbol{u}) \tag{2.13}$$

is a bounded continuous *operator from* $\boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$ *into* $\boldsymbol{L}^{p'}(\Omega,\mathbb{R}^m)$.

Remark 2.4 *In consideration of* **Remark 2.2** *it results that if* $p > 2$ *then the operator*

$$\boldsymbol{u} \mapsto -\operatorname{div} \boldsymbol{S}(\boldsymbol{u},\nabla\boldsymbol{u}) - \boldsymbol{b}(\boldsymbol{u},\nabla\boldsymbol{u}) \tag{2.14}$$

from $\boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$ *into* $\boldsymbol{X}(\Omega,\mathbb{R}^m)$ *is a* continuous *operator and in view of* (1.3) *it follows that, for* $p>2$, *the equation* (2.1) *makes sense for* $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega,\mathbb{R}^m)$.

We point out that the operator (2.14) *determines in a unique way the bounded and continuous operator*

$$\begin{cases} \boldsymbol{u} \mapsto \Lambda(\boldsymbol{u}) := A(\boldsymbol{u}) - B(\boldsymbol{u}) \in X(\Omega,\mathbb{R}^m), \quad \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m), p>2, \\ \langle \boldsymbol{v}, \Lambda(\boldsymbol{u}) \rangle = \int\limits_\Omega [\boldsymbol{S}(\boldsymbol{u},\nabla\boldsymbol{u}) \cdot \nabla\boldsymbol{v} - \boldsymbol{b}(\boldsymbol{u},\nabla\boldsymbol{u}) \cdot \boldsymbol{v}] \mathrm{d}\boldsymbol{x}, \quad \boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m). \end{cases} \tag{2.15}$$

3. WEAK SOLUTIONS OF THE DIRICHLET PROBLEM FOR THE SYSTEM (2.1)

In the conditions of the preceding Section we have in view to prove the existence of weak solutions $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$, $p>2$, of the Dirichlet problem

(P)
$$\begin{cases} -\operatorname{div} \boldsymbol{S}(\nabla\boldsymbol{u},\nabla\boldsymbol{u}) - \boldsymbol{b}(\boldsymbol{u},\nabla\boldsymbol{u}) = \boldsymbol{f} \ \text{ in } \Omega, \\ \boldsymbol{u} = \boldsymbol{u}_0 \ \text{ on } \partial\Omega, \end{cases}$$

where $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega,\mathbb{R}^m)$ and $\boldsymbol{u}_0 \in \boldsymbol{W}^{1/p',p}(\partial\Omega,\mathbb{R}^m)$.

The function $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$ is called a *weak solution* to the problem (P) if $\boldsymbol{u}$ is the solution of the variational problem

(VP)
$$\Lambda(\boldsymbol{u}) = \boldsymbol{f}, \quad \boldsymbol{u} - \boldsymbol{g} \in \boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m),$$

where the operator Λ is defined by (2.15) and $\boldsymbol{g} \in \boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$ is a mapping having the trace $\boldsymbol{u}_0$ on $\partial\Omega$, $\operatorname{tr}\boldsymbol{g} = \boldsymbol{u}_0$ (such a mapping does exist [2], [3], [9]). The variational problem (VP) comes back to the variational problem

$$\Lambda^g(\boldsymbol{u}) := \Lambda(\boldsymbol{g}+\boldsymbol{u}) = \boldsymbol{f}, \ \ \boldsymbol{u} \in \boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m), \tag{3.1}$$

which is equivalent to the problem of finding $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$ such that

$$\langle \boldsymbol{v}, \Lambda^g(\boldsymbol{u}) - \boldsymbol{f} \rangle := \int\limits_\Omega \{\nabla\boldsymbol{v} \cdot \boldsymbol{S}(\boldsymbol{g}+\boldsymbol{u},\nabla(\boldsymbol{g}+\boldsymbol{u})) - \boldsymbol{v} \cdot [\boldsymbol{b}(\boldsymbol{g}+\boldsymbol{u},\nabla(\boldsymbol{g}+\boldsymbol{u})) - \boldsymbol{f}]\} \mathrm{d}\boldsymbol{x} = 0, \tag{3.1'}$$

for every $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$.

Further we are going to use the following

Lemma 3.1 For every $\boldsymbol{g} \in \boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$ and $\boldsymbol{u},\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$ we have

$$\langle \boldsymbol{u}-\boldsymbol{v}, \Lambda^g(\boldsymbol{u}) - \Lambda^g(\boldsymbol{v}) \rangle = L_0(\boldsymbol{g},\boldsymbol{u},\boldsymbol{v}) + L_1(\boldsymbol{g},\boldsymbol{u},\boldsymbol{v}), \tag{3.2}$$

Where

$$\begin{cases} L_0(\boldsymbol{g},\boldsymbol{u},\boldsymbol{v}) = \int\limits_0^1 \mathrm{d}t \int\limits_\Omega \nabla\boldsymbol{h} \cdot \left[\dfrac{\partial\boldsymbol{S}}{\partial\boldsymbol{p}}(\boldsymbol{g}+\boldsymbol{w},\nabla(\boldsymbol{g}+\boldsymbol{w}))\boldsymbol{h} + \dfrac{\partial\boldsymbol{S}}{\partial\boldsymbol{P}}(\boldsymbol{g}+\boldsymbol{w},\nabla(\boldsymbol{g}+\boldsymbol{w}))\nabla h \right] \mathrm{d}\boldsymbol{x}, \\ L_1(\boldsymbol{g},\boldsymbol{u},\boldsymbol{v}) = \int\limits_0^1 \mathrm{d}t \int\limits_\Omega \boldsymbol{h} \cdot \left[\dfrac{\partial\boldsymbol{b}}{\partial\boldsymbol{p}}(\boldsymbol{g}+\boldsymbol{w},\nabla(\boldsymbol{g}+\boldsymbol{w}))\boldsymbol{h} + \dfrac{\partial\boldsymbol{b}}{\partial\boldsymbol{P}}(\boldsymbol{g}+\boldsymbol{w},\nabla(\boldsymbol{g}+\boldsymbol{w}))\nabla h \right] \mathrm{d}\boldsymbol{x}, \end{cases} \tag{3.3}$$

and $\boldsymbol{w} = \boldsymbol{v} + t\boldsymbol{h}$, $\boldsymbol{h} = \boldsymbol{u} - \boldsymbol{v}$.

PROOF: From (2.15) we obtain

$$\langle \boldsymbol{u}-\boldsymbol{v}, \Lambda^{g}(\boldsymbol{u})-\Lambda^{g}(\boldsymbol{v})\rangle =$$

$$=\int_\Omega \nabla \boldsymbol{h}\cdot[\boldsymbol{S}(\boldsymbol{g}+\boldsymbol{u},\nabla(\boldsymbol{g}+\boldsymbol{u}))-\boldsymbol{S}(\boldsymbol{g}+\boldsymbol{v},\nabla(\boldsymbol{g}+\boldsymbol{v}))]\mathrm{d}\boldsymbol{x}-\int_\Omega \boldsymbol{h}\cdot[\boldsymbol{b}(\boldsymbol{g}+\boldsymbol{u},\nabla(\boldsymbol{g}+\boldsymbol{u}))-\boldsymbol{b}(\boldsymbol{g}+\boldsymbol{v},\nabla(\boldsymbol{g}+\boldsymbol{v}))]\mathrm{d}\boldsymbol{x}=$$

$$=\int_\Omega[\nabla \boldsymbol{h}\cdot\int_0^1\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t}(\boldsymbol{g}+\boldsymbol{w},\nabla(\boldsymbol{g}+\boldsymbol{w}))\mathrm{d}t]\mathrm{d}\boldsymbol{x}-\int_\Omega[\boldsymbol{h}\cdot\int_0^1\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}(\boldsymbol{g}+\boldsymbol{w},\nabla(\boldsymbol{g}+\boldsymbol{w}))\mathrm{d}t]\mathrm{d}\boldsymbol{x}$$

where $\boldsymbol{w}=\boldsymbol{v}+t\boldsymbol{h}$, $\boldsymbol{h}=\boldsymbol{u}-\boldsymbol{v}$. Taking into account that $\boldsymbol{S}(\boldsymbol{x},\cdot,\cdot)$ and $\boldsymbol{b}(\boldsymbol{x},\cdot,\cdot)$ are Fréchet differentiable and applying the Chain Rule we get (3.2).

4. AN EXISTENCE RESULT OF THE PROBLEM (P)

THEOREM 4.1 *If for every* $\boldsymbol{u}$, $\boldsymbol{h}\in \boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$ *and* $(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P})\in\Omega\times\mathbb{R}^m\times\mathbb{M}_{m\times n}$ *we have*

$$(\mathbf{H}_1)\qquad \begin{cases}\displaystyle\int_0^1\mathrm{d}t\int_\Omega \nabla\boldsymbol{h}\cdot\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{P}}(\boldsymbol{u}+t\boldsymbol{h},\nabla(\boldsymbol{u}+t\boldsymbol{h}))\nabla\boldsymbol{h}\mathrm{d}\boldsymbol{x}\ge c_0\,\|\boldsymbol{h}\|_{1,p}^{p},\\ \displaystyle\int_0^1\mathrm{d}t\int_\Omega \nabla\boldsymbol{h}\cdot\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{p}}(\boldsymbol{u}+t\boldsymbol{h},\nabla(\boldsymbol{u}+t\boldsymbol{h}))\boldsymbol{h}\mathrm{d}\boldsymbol{x}\ge 0\end{cases}$$

and

$$(\mathbf{H}_2)\qquad \begin{cases}\displaystyle\int_0^1\mathrm{d}t\int_\Omega \boldsymbol{h}\cdot\frac{\partial \boldsymbol{b}}{\partial \boldsymbol{p}}(\boldsymbol{u}+t\boldsymbol{h},\nabla(\boldsymbol{u}+t\boldsymbol{h}))\boldsymbol{h}\mathrm{d}\boldsymbol{x}\le 0\\ \displaystyle\left|\frac{\partial \boldsymbol{b}}{\partial \boldsymbol{P}}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P})\right|\le c_1(1+|\boldsymbol{p}|^{q-1}+|\boldsymbol{P}|^{q-1}),\end{cases}$$

where $q\in(1,p-1)=(1,p/p')$, *and* $c_0>0$, $c_1>0$ *are constants independent of* $\boldsymbol{u}$, $\boldsymbol{h}$ *and* $(\boldsymbol{x},\boldsymbol{p},\boldsymbol{P})$, *then the bounded and continuous operator*

$$\begin{cases}\boldsymbol{u}\mapsto\Lambda^{g}(\boldsymbol{u})\in \boldsymbol{W}^{-1,p}(\Omega,\mathbb{R}^m), & \boldsymbol{u}\in \boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m),\\ \langle\boldsymbol{v},\Lambda^{g}(\boldsymbol{u})\rangle=\displaystyle\int_\Omega[\nabla\boldsymbol{v}\cdot\boldsymbol{S}(\boldsymbol{g}+\boldsymbol{u},\nabla(\boldsymbol{g}+\boldsymbol{u}))-\boldsymbol{v}\cdot\boldsymbol{b}(\boldsymbol{g}+\boldsymbol{u},\nabla(\boldsymbol{g}+\boldsymbol{u}))]\mathrm{d}\boldsymbol{x}, & \boldsymbol{v}\in \boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m),\end{cases}\tag{4.1}$$

is a Gårding coercive *operator*.

PROOF: **A**. ***The operator (4.1) is a Gårding operator.*** In view of imbeddings (1.1) and (1.2) we can chose $V=\boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m)$ and $U=\boldsymbol{L}^p(\Omega,\mathbb{R}^m)$ in Def. 1.1 of Gårding operators. In this definition we take [10]

$$(\boldsymbol{u},\boldsymbol{v})\mapsto F(\boldsymbol{u},\boldsymbol{v}):=\Lambda^{g}(\boldsymbol{u})+\boldsymbol{0}(\boldsymbol{v})=\Lambda^{g}(\boldsymbol{u})\in \boldsymbol{W}^{-1,p'}(\Omega,\mathbb{R}^m),\quad \boldsymbol{u},\boldsymbol{v}\in \boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m),\tag{4.2}$$

where $\boldsymbol{0}$ is the null operator. With this choice, the condition (iii) in Def. 1.1 is trivially satisfied because $F(\boldsymbol{v},\boldsymbol{u}_n)-F(\boldsymbol{v},\boldsymbol{u})=0$ for every $\boldsymbol{v}\in \boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m)$. The condition (i) of Def. 1.1 is fulfilled since $F(\boldsymbol{u}+t\boldsymbol{v},\boldsymbol{w})=\Lambda^{g}(\boldsymbol{u}+t\boldsymbol{v})$ for every $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in \boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m)$ and $t\in\mathbb{R}$, and the real function

$$t\mapsto\langle\boldsymbol{v},\Lambda^{g}(\boldsymbol{u}+t\boldsymbol{v})\rangle=\int_\Omega[\nabla\boldsymbol{v}\cdot\boldsymbol{S}(\boldsymbol{u}+t\boldsymbol{v},\nabla(\boldsymbol{u}+t\boldsymbol{v}))-\boldsymbol{v}\cdot\boldsymbol{b}(\boldsymbol{u}+t\boldsymbol{v},\nabla(\boldsymbol{u}+t\boldsymbol{v}))]\mathrm{d}\boldsymbol{x}\in\mathbb{R},t\in\mathbb{R}$$

is continuous in consideration of condition $(\mathbf{I})_b$ and $(\mathbf{II})_b$. Consequently, to prove that Λ^g is a Gårding operator we have only to show that, with F given by (4.2), the condition (ii) of Def. 1.1 is verified.

Taking into account $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ we get

$$L_0(\boldsymbol{g},\boldsymbol{u},\boldsymbol{v}) \geq c_0 \, \| \boldsymbol{u}-\boldsymbol{v} \|_{1,p}^p , \tag{4.3}$$

$$-L_1(\boldsymbol{g},\boldsymbol{u},\boldsymbol{v}) \leq c_1 \int_\Omega |\boldsymbol{h}| |\nabla \boldsymbol{h}| \, \mathrm{d}\boldsymbol{x} \int_0^1 [1+|\boldsymbol{g}+\boldsymbol{w}|^{q-1}+|\nabla \boldsymbol{g}+\nabla \boldsymbol{w}|^{q-1}] \mathrm{d}t \leq$$

$$\leq c_1 \int_0^1 \mathrm{d}t \int_\Omega |\boldsymbol{h}| |\nabla \boldsymbol{h}| \{1+2^{q-1}[(|\boldsymbol{g}|^{q-1}+|\nabla \boldsymbol{g}|^{q-1})+(|\boldsymbol{v}+t\boldsymbol{h}|^{q-1}+|\nabla(\boldsymbol{v}+t\boldsymbol{h})|^{q-1})]\} d\boldsymbol{x}. \tag{4.4}$$

Using some elementary results from the theory of $L^p(\Omega)$ spaces ([1], [2], [7]), taking into account that $p>2$, and $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain we obtain: $a)$ Because $\boldsymbol{h} \in \boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$ it follows that $|\boldsymbol{h}|$, $|\nabla \boldsymbol{h}| \in L^p(\Omega) \subset L^2(\Omega)$ and therefore

$$\int_\Omega |\boldsymbol{h}| |\nabla \boldsymbol{h}| \, d\boldsymbol{x} \leq \| \boldsymbol{h} \|_2 \| \nabla \boldsymbol{h} \|_2 \leq \text{const.} \| \boldsymbol{h} \|_p \| \nabla \boldsymbol{h} \|_p \tag{4.5}$$

since $\|\cdot\|_2 \leq \text{const.}\|\cdot\|_p$. $b)$ Let us point out the implications:

$$q \in (1, p-1) \Rightarrow 0 < p(q-1)/(p-2) < p \Rightarrow L^p(\Omega) \subset L^s(\Omega), \quad s = p(q-1)/(p-2),$$
$$\boldsymbol{g} \in \boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m) \Rightarrow |\boldsymbol{g}| \in L^p(\Omega) \subset L^s(\Omega) \Rightarrow |\boldsymbol{g}|^{q-1} \in L^{p/(p-2)}(\Omega).$$

As $p^{-1}+p^{-1}+(\dfrac{p}{p-2})^{-1}=1$, by virtue of generalized Hölder inequality ([2], [7]), it results that $|\boldsymbol{h}| |\nabla \boldsymbol{h}| |\boldsymbol{g}|^{q-1} \in L^1(\Omega)$ and

$$\int_\Omega |\boldsymbol{h}| |\nabla \boldsymbol{h}| |\boldsymbol{g}|^{q-1} \, d\boldsymbol{x} \leq \| \boldsymbol{h} \|_p \| \nabla \boldsymbol{h} \|_p \| |\boldsymbol{g}|^{q-1} \|_{p/(p-2)} .$$

On the other hand we have $\| |\boldsymbol{g}|^{q-1} \|_{p/(p-2)} = \| \boldsymbol{g} \|_s^{q-1} \leq \| \boldsymbol{g} \|_p^{q-1}$, whereof we obtain

$$\int_\Omega |\boldsymbol{h}| |\nabla \boldsymbol{h}| |\boldsymbol{g}|^{q-1} \, d\boldsymbol{x} \leq \| \boldsymbol{h} \|_p \| \nabla \boldsymbol{h} \|_p \| |\boldsymbol{g}|^{q-1} \|_p \leq \text{const.} \| \boldsymbol{h} \|_p \| \boldsymbol{h} \|_{1,p} . \tag{4.6}$$

$c)$ From the implications $\boldsymbol{g} \in \boldsymbol{W}^{1,p}(\Omega,R^m) \Rightarrow |\nabla \boldsymbol{g}| \in L^p(\Omega) \subset L^s(\Omega) \Rightarrow \nabla \boldsymbol{g}| \in L^s(\Omega) \Rightarrow$

$|\nabla \boldsymbol{g}|^{q-1} \in L^{p/(p-2)}(\Omega)$, $\boldsymbol{h} \in W^{1,p}(\Omega,\mathbb{R}^m) \Rightarrow |\boldsymbol{h}|, |\nabla \boldsymbol{h}| \in L^p(\Omega)$, and from $p^{-1}+p^{-1}+(\dfrac{p}{p-2})^{-1}=1$ it results that $|\boldsymbol{h}| |\nabla \boldsymbol{h}| |\nabla \boldsymbol{g}|^{q-1} \in L^1(\Omega)$ and

$$\int_\Omega |\boldsymbol{h}| |\nabla \boldsymbol{h}| |\nabla \boldsymbol{g}|^{q-1} \, d\boldsymbol{x} \leq \| \boldsymbol{h} \|_p \| \nabla \boldsymbol{h} \|_p \| \nabla \boldsymbol{g} \|_{p/(p-2)} .$$

As $\| |\nabla \boldsymbol{g}|^{q-1} \|_{p/(p-2)} = \| \nabla \boldsymbol{g} \|_s^{q-1} \leq \| \nabla \boldsymbol{g} \|_p^{q-1}$ it follows

$$\int_\Omega |\boldsymbol{h}| |\nabla \boldsymbol{h}| |\nabla \boldsymbol{g}|^{q-1} \, d\boldsymbol{x} \leq \| \boldsymbol{h} \|_p \| \nabla \boldsymbol{h} \|_p \| \nabla \boldsymbol{g} \|_p \leq \text{const.} \| \boldsymbol{h} \|_p \| \boldsymbol{h} \|_{1,p} . \tag{4.7}$$

d) Similarly with (4.6) and (4.7) we obtain

$$\int_\Omega |\boldsymbol{h}||\nabla\boldsymbol{h}||\boldsymbol{v}+t\boldsymbol{h}|^{q-1}\,d\boldsymbol{x} \le \text{const.}\|\boldsymbol{h}\|_p\|\boldsymbol{h}\|_{1,p}\|\boldsymbol{v}+t\boldsymbol{h}\|_p^{q-1}, \tag{4.8}$$

$$\int_\Omega |\boldsymbol{h}||\nabla\boldsymbol{h}||\nabla(\boldsymbol{v}+t\boldsymbol{h})|^{q-1}\,d\boldsymbol{x} \le \text{const.}\|\boldsymbol{h}\|_p\|\boldsymbol{h}\|_{1,p}\|\nabla(\boldsymbol{v}+t\boldsymbol{h})\|_p^{q-1}. \tag{4.9}$$

From $(4.4)_2$ and (4.5)–(4.9) we have

$$-L_1(\boldsymbol{g},\boldsymbol{u},\boldsymbol{v}) \le \text{const.}\|\boldsymbol{h}\|_p\|\boldsymbol{h}\|_{1,p}(1+\|\boldsymbol{v}+\theta\boldsymbol{h}\|_p^{q-1}+\|\nabla(\boldsymbol{v}+\theta\boldsymbol{h})\|_p^{q-1}), \tag{4.10}$$

and, on the other hand

$$\begin{cases}\|\boldsymbol{v}+\theta\boldsymbol{h}\|_p \le \|\boldsymbol{v}+\theta\boldsymbol{h}\|_{1,p} \le \|\boldsymbol{u}\|_{1,p}+2\|\boldsymbol{v}\|_{1,p}, \\ \|\nabla(\boldsymbol{v}+\theta\boldsymbol{h})\|_p \le \|\boldsymbol{v}+\theta\boldsymbol{h}\|_{1,p} \le \|\boldsymbol{u}\|_{1,p}+2\|\boldsymbol{v}\|_{1,p},\end{cases} \quad \theta\in(0,1). \tag{4.11}$$

In view of the dense imbedding (1.2) and (4.11) we have

$$\|\boldsymbol{v}+\theta\boldsymbol{h}\|_p^{q-1} \le \text{const.}\,r^{q-1}, \quad \|\nabla(\boldsymbol{v}+\theta\boldsymbol{h})\|_p^{q-1} \le \text{const.}\,r^{q-1}, \tag{4.12}$$

for every $\boldsymbol{u},\boldsymbol{v}\in B_r(\boldsymbol{0})=\{\boldsymbol{u}\in \boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m):\|\boldsymbol{u}\|_{1,p}<r\}$. From (4.10) and (4.12) it results

$$-L_1(\boldsymbol{g},\boldsymbol{u},\boldsymbol{v}) \le \|\boldsymbol{h}\|_p\|\boldsymbol{h}\|_{1,p}(a_1+a_2r^{q-1}), \quad \boldsymbol{h}=\boldsymbol{u}-\boldsymbol{v}, \ \boldsymbol{u},\boldsymbol{v}\in B_r(\boldsymbol{0}), \tag{4.13}$$

where $a_1>0$ and $a_2>0$ are constants depending on Ω, p, and $\boldsymbol{g}$. By using a variant of the Young inequality [2] we get

$$\begin{cases}\|\boldsymbol{h}\|_p\|\boldsymbol{h}\|_{1,p} \le \varepsilon\|\boldsymbol{h}\|_{1,p}^p + c(\varepsilon)\|\boldsymbol{h}\|_p^{p'}, \\ \|\boldsymbol{h}\|_p\|\boldsymbol{h}\|_{1,p}\,r^{q-1} \le \varepsilon\|\boldsymbol{h}\|_{1,p}^p + c(\varepsilon)\|\boldsymbol{h}\|_p^{p'}\,r^{(q-1)p'},\end{cases} \tag{4.14}$$

where $\varepsilon>0$ is an arbitrary constant, $c(\varepsilon)=\varepsilon^{-1/(p-1)}$, and $\boldsymbol{h}=\boldsymbol{u}-\boldsymbol{v}$. Therefore, from (4.13), (4.14) we have

$$-L_1(\boldsymbol{g},\boldsymbol{u},\boldsymbol{v}) \le (a_1+a_2)\varepsilon\|\boldsymbol{h}\|_{1,p}^p + (a_1+a_2r^{(q-1)p'})c(\varepsilon)\|\boldsymbol{h}\|_p^{p'}, \tag{4.15}$$

whereof, in view of (4.3), from (3.2) it results

$$\left\langle \boldsymbol{u}-\boldsymbol{v},\Lambda^g(\boldsymbol{u})-\Lambda^g(\boldsymbol{v})\right\rangle \ge c_0\|\boldsymbol{h}\|_{1,p}^p-(a_1+a_2)\varepsilon\|\boldsymbol{h}\|_{1,p}^p-(a_1+a_2r^{(q-1)p'})c(\varepsilon)\|\boldsymbol{h}\|_p^{p'}.$$

If in this inequality we take $\varepsilon>0$ sufficiently small, it follows that for every $\boldsymbol{u}$, $\boldsymbol{v}\in B_r(\boldsymbol{0})$ we have

$$\left\langle \boldsymbol{u}-\boldsymbol{v},\Lambda^g(\boldsymbol{u})-\Lambda^g(\boldsymbol{v})\right\rangle \ge b_0\|\boldsymbol{u}-\boldsymbol{v}\|_{1,p}^p-(b_1+b_2r^{(q-1)p'})\|\boldsymbol{u}-\boldsymbol{v}\|_p^{p'}, \tag{4.16}$$

where $b_0>0$, $b_1>0$, and $b_2>0$ are constant. Thus we proved that Λ^g is a Gårding operator for every $\boldsymbol{g}\in\boldsymbol{W}^{1,p}(\Omega,\mathbb{R}^m)$ since (4.16) implies

$$\left\langle \boldsymbol{u}-\boldsymbol{v},\Lambda^g(\boldsymbol{u})-\Lambda^g(\boldsymbol{v})\right\rangle \ge -\gamma(r,\|\boldsymbol{u}-\boldsymbol{v}\|_p), \qquad \boldsymbol{u},\boldsymbol{v}\in B_r(\boldsymbol{0}), \tag{4.17}$$

where $\gamma(x,y)=(b_1+b_2x^{(q-1)p'})y^{p'}$, $x\ge 0$, $y\ge 0$, satisfies $\lim_{\theta\downarrow 0}\theta^{-1}\gamma(x,\theta y)=0$, $\forall x$, $y>0$.

B. ***The operator (4.1) is coercive***. By taking $\boldsymbol{v}=0$ in (3.2) we obtain

$$\left\langle \boldsymbol{u}, \Lambda^g(\boldsymbol{u})\right\rangle = L_0(\boldsymbol{g},\boldsymbol{u},\boldsymbol{0}) + L_1(\boldsymbol{g},\boldsymbol{u},\boldsymbol{0}) + \left\langle \boldsymbol{u}, \Lambda^g(\boldsymbol{0})\right\rangle, \tag{4.18}$$

From (3.3) with $\boldsymbol{v}=0$ and hypotheses $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ we obtain

$$\begin{cases} L_0(\boldsymbol{g},\boldsymbol{u},\boldsymbol{0}) & \geq c_0 \,\|\, \boldsymbol{u} \,\|_{1,p}^p, \\ -L_1(\boldsymbol{g},\boldsymbol{u},\boldsymbol{0}) & \leq \int\limits_0^1 dt \int\limits_\Omega |\,\boldsymbol{u}\,\|\,\nabla\boldsymbol{u}\,|\,\{1+2^{q-1}[|\,\boldsymbol{g}\,|^{q-1}+|\,\nabla\boldsymbol{g}\,|^{q-1}+t^{q-1}(|\,\boldsymbol{u}\,|^{q-1}+|\,\nabla\boldsymbol{u}\,|^{q-1})]\}d\boldsymbol{x}. \end{cases} \tag{4.19}$$

If in estimations (4.5)–(4.9) we take $\boldsymbol{v}=0$ and use the Young inequality as in (4.4) we obtain

$$-L_1(\boldsymbol{g},\boldsymbol{u},\boldsymbol{0}) \leq A_1\varepsilon\|\boldsymbol{u}\|_{1,p}^{p'} + A_2c(\varepsilon)\|\boldsymbol{u}\|_{1,p}^{p'} + A_3c(\varepsilon)\|\boldsymbol{u}\|_{1,p}^{qp'}, \tag{4.20}$$

where $A_1>0$, $A_2>0$, $A_3>0$ are constants depending on $\Omega, q, \boldsymbol{g}$, and $\varepsilon>0$ is an arbitrary constant.

Applying succesively Hölder and Young inequalities we have

$$\begin{aligned} |<\boldsymbol{u},\Lambda^g(\boldsymbol{0})>| &\leq \|\, \boldsymbol{S}(\boldsymbol{g},\nabla\boldsymbol{g})\,\|_{p'} \|\,\nabla\boldsymbol{u}\,\|_p + \|\,\boldsymbol{b}(\boldsymbol{g},\nabla\boldsymbol{g})\,\|_{p'}\|\,\boldsymbol{u}\,\|_p \leq \\ &\leq \varepsilon_0[\|\,\boldsymbol{S}(\boldsymbol{g},\nabla\boldsymbol{g})\,\|_{p'}^p + \|\,\boldsymbol{b}(\boldsymbol{g},\nabla\boldsymbol{g})\,\|_{p'}^p] + c(\varepsilon_0)[\|\,\boldsymbol{u}\,\|_p^{p'} + \|\,\nabla\boldsymbol{u}\,\|_p^{p'}] \leq \\ &\leq B_1 + B_2\,\|\,\boldsymbol{u}\,\|_{1,p}^{p'}, \end{aligned} \tag{4.21}$$

where $B_1>0$ and $B_2>0$ are constants with evident dependence on $\boldsymbol{S}$ and $\boldsymbol{b}$. From (4.18)–(4.21) we obtain

$$\left\langle \boldsymbol{u},\Lambda^g(\boldsymbol{u})\right\rangle \geq (c_0 - A_1\varepsilon)\,\|\,\boldsymbol{u}\,\|_{1,p}^p - (A_2c(\varepsilon)+B_2)\,\|\,\boldsymbol{u}\,\|_{1,p}^{p'} - A_3c(\varepsilon)\,\|\,\boldsymbol{u}\,\|_{1,p}^{qp'} - B_1, \tag{4.22}$$

where $\varepsilon>0$ is an arbitrary constant. If we take $\varepsilon>0$ sufficiently small in (4.22) it results

$$\left\langle \boldsymbol{u},\Lambda^g(\boldsymbol{u})\right\rangle \geq C_0\,\|\,\boldsymbol{u}\,\|_{1,p}^p - C_1\,\|\,\boldsymbol{u}\,\|_{1,p}^{p'} - C_2\,\|\,\boldsymbol{u}\,\|_{1,p}^{qp'} - B_1, \tag{4.23}$$

from where we get

$$\|\,\boldsymbol{u}\,\|_{1,p}^{-1}\left\langle \boldsymbol{u},\Lambda^g(\boldsymbol{u})\right\rangle \geq \|\,\boldsymbol{u}\,\|_{1,p}^{p-1}[C_0 - C_1\,\|\,\boldsymbol{u}\,\|_{1,p}^{p'-p} - C_2\,\|\,\boldsymbol{u}\,\|_{1,p}^{qp'-p} - B_1\,\|\,\boldsymbol{u}\,\|_{1,p}^{-p}] \tag{4.24}$$

for every $\boldsymbol{u}\in \boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m)$. Since $p-1>1$, $p'-p<0$, and $qp'<p$ we obtain

$$\|\,\boldsymbol{u}\,\|_{1,p}^{-1}\left\langle \boldsymbol{u},\Lambda^g(\boldsymbol{u})\right\rangle \to \infty \text{ as } \|\,\boldsymbol{u}\,\|_{1,p}\to\infty, \qquad \boldsymbol{u}\in\boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m),$$

and the theorem is proved (see (1.4)).

Remark 4.1 *Because a bounded Gårding operator is pseudomonotone* [11], *it follows the implication* [8]

$$\left.\begin{array}{c} \boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \text{ in } \boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m) \\ \text{and} \\ \limsup\limits_{n\to\infty}\left\langle \boldsymbol{u}_n-\boldsymbol{u},\Lambda^g(\boldsymbol{u}_n)\right\rangle \leq 0 \end{array}\right\} \Rightarrow \liminf_{n\to\infty}\left\langle \boldsymbol{u}_n-\boldsymbol{v},\Lambda^g(\boldsymbol{u}_n)\right\rangle \geq \left\langle \boldsymbol{u}-\boldsymbol{v},\Lambda^g(\boldsymbol{u})\right\rangle,$$

for every $\boldsymbol{v}\in\boldsymbol{W}_0^{1,p}(\Omega,\mathbb{R}^m)$.

From theorems 1.1 and 4.1 we obtain the desired existence result.

THEOREM 4.2 *If* $\Omega \subset \mathbb{R}^n$ *is a Lipschitz bounded domain,* $p > 2$ *, and the mappings (2.2) and (2.3) satisfy the restrictions* $(\mathbf{I})$ *and* $(\mathbf{II})$ *of Section 2 and the hypotheses* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *in theorem 4.1 then, for every pair* $(\boldsymbol{f}, \boldsymbol{u}_0) \in \boldsymbol{L}^p(\Omega, \mathbb{R}^m) \times \boldsymbol{W}^{1/p', p}(\partial\Omega, \mathbb{R}^m)$ *, there exists at least one weak solution* $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega, \mathbb{R}^m)$ *of the problem (P).*

Remark 4.2 *From the proof of lemma 3.1 and hypothesis* $(\mathbf{H}_1)_1$ *it results that the operator*

$$\boldsymbol{u} \mapsto A^{\boldsymbol{g}}(\boldsymbol{u}) := A(\boldsymbol{g} + \boldsymbol{u}) \in \boldsymbol{W}^{-1,p}(\Omega, \mathbb{R}^m), \qquad \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega, \mathbb{R}^m),$$

defined by (2.9), is a p-coercive, and consequently a strongly monotone operator ([9]) for every $\boldsymbol{g} \in \boldsymbol{W}^{1,p}(\Omega, \mathbb{R}^m)$ *, i.e.*

$$\left\langle \boldsymbol{u} - \boldsymbol{v}, A^{\boldsymbol{g}}(\boldsymbol{u}) - A^{\boldsymbol{g}}(\boldsymbol{v}) \right\rangle \geq c_0 \left\| \boldsymbol{u} - \boldsymbol{v} \right\|_{1,p}^{p}, \qquad c_0 > 0, \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega, \mathbb{R}^m).$$

Remark 4.3 *If the mapping (2.2) is independent of* $\boldsymbol{u}$ *and* $m = n \geq 1$ *then the system (2.1) is a quasilinear differential system of finite n-dimensional elastostatics type. In [4] we obtained some existence results of the weak solutions to the Dirichlet problem for such a system in three dimensions.*

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