THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 8, Number 2/2007, pp. 000-000

COMPARISONS OF G/G/1 QUEUES

Laszlo LAKATOS and Gheorghiță ZBĂGANU

Lorand Eötvos University, Budapest, E-mail: lakatos@compalg.inf.elte.hu
"Gheorghe Mihoc – Caius Iacob" Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy
Casa Academiei Române, Calea 13 Septembrie no. 13, 050711 Bucharest, Romania. E-mail:zbagang@fmi.unibuc.ro

We discuss the following problem: if we have two G/G/1 queues, which of them is better? There are several possibilities to understand the word "better". In our approach "better" means that the waiting time of the *n*th customer is stochastically smaller in the first queue than in the second one.

1. NOTATION AND STATEMENT OF THE PROBLEM

A ***queueing system*** (or, shortly, a ***queue***) works as follows: there exist ***customers*** which want a ***service*** provided by some server. The ***service time*** is the time needed by the server to do its job. If a customer comes and there is at least another customer which is served, we say that the system is ***busy***; otherwise, it is ***free***. The ***discipline*** of the queue is the rule according to which the customers are served. The simplest is *FCFS* (First Come, First Served) which is also denoted by some authors by *FIFO* (First In, First Out).

We ***know*** the queue when we know the arrival times and the service times. Let us denote by $(t_n)_{n\geq 1}$ the sequence of the arrival times and by $(S_n)_n$ the sequence of the service times. Thus t_n is the moment when the n'th customer arrives in line and S_n is the time needed by the server to satisfy him.

We can safely assume that the sequence $(t_n)_n$ is ***non-decreasing*** and that $S_n > 0$ – there is no instantaneous service. We denote such a queuing system by $[(S_n)_n ; (t_n)_n]$.

The time when the nth customer is served will be denoted by τ_n. Remark that

$$\tau_n = t_n + W_n + S_n , \tag{1.1}$$

where W_n is the time spent by the n'th customer from arrival until his service begins. This is the ***waiting time*** of the nth customer.

The result below is well known.

Proposition 1.1. The waiting times satisfy the recurrence

$$W_{n+1} = (W_n + S_n - (t_{n+1} - t_n))_+ \ \forall\ n \geq 1,\ \ W_1 = 0. \tag{1.2}$$

Proof. Of course, the first customer has no need to wait, thus $W_1 = 0$. The $(n+1)$th customer arrives at the moment t_{n+1}. The service of the nth customer is finished at τ_n. If $\tau_n \leq t_{n+1}$ the system is free, thus there is no waiting hence $W_{n+1} = 0$. Otherwise, he has to wait from the arrival – i.e. from t_{n+1} – until τ_n. The waiting time will be in that case $\tau_n - t_{n+1}$. Therefore $W_{n+1} = (\tau_n - t_{n+1})_+$ □

It is usual to denote the ***interarrival times*** $t_{n+1} - t_n$ by T_n . In that case the recurrence (1.2) becomes nicer if written as

$$W_{n+1} = (W_n + S_n - T_n)_+ \ \forall\ n \geq 1,\ \ W_1 = 0. \tag{1.3}$$

What maters from the point of view of waiting times are only the differences $\xi_n = S_n - T_n$. So, if we are interested only in their study, the recurrence becomes

$$W_{n+1} = (W_n + \xi_n)_+ \ \forall\ n \geq 1,\ \ W_1 = 0. \tag{1.4}$$

Now, it is obvious that the sequence $(\tau_n)_n$ is ***increasing.*** Indeed, $\tau_{n+1} - \tau_n = t_{n+1} + W_{n+1} + S_{n+1} - t_n - W_n - S_n = T_n + (W_n + S_n - T_n)_+ - W_n + S_{n+1} - S_n \geq T_n + (W_n + S_n - T_n) - W_n + S_{n+1} - S_n = S_{n+1} > 0$.

We can compare two queuing systems in many ways. From the point of view of the customers it is convenient to consider the following

Definition. *The system* $[(S_n)_n\ ;\ (t_n)_n]$ *is absolutely better than* $[(S'_n)_n\ ;\ (t'_n)_n]$ *iff* $W_n \leq W'_n\ \forall\ n \geq 1$ *where* W'_n *are the waiting times in the second queuing system.*

We shall only deal with this meaning of the word "better". We shall assume the simplest situation, namely, the random sequences $(S_n)_n$ and $(T_n)_n$ are i.i.d. **and independent.** In that case the number of customers which arrive in line in the interval $[0,t]$ will be a renewal process. (However, this is not true in general about the number of customers served in the interval $[0, t]$ due to obvious reasons: the random variables $(\tau_{n+1} - \tau_n)_{n\geq 1}$ have no reason to be i.i.d.: just look at their definition).

Clearly, such a queuing system is uniquely determined by the distributions of the service times S_n and the interarrival times T_n. A general queuing system is usually abbreviated as *GI/GI/*1 *FCFS*. ("*G"* stands for "general", "1" is the number of servers and *FCFS* is the discipline of the queue: first come first served).

Important notation. A *GI/GI/*1 *FCFS* queuing system where the distribution of S_n is F and the distribution of T_n is G, will be denoted by $\{F,G\}$ or, alternatively, by $\{S,T\}$. If F = Exponential(a) and G = Exponential (b), then the classical notation for such a queuing system is *M/M/*1 (in this case the arrival process is Markovian and the numbers of customers in line is Markovian, too). We shall denote such a *M/M/*1 queuing system simply by $\{a,b\}$. Of course $\mathrm{E}S_n = 1/a$ and $\mathrm{E}T_n = 1/b$. The ratio $\mathrm{E}S_n / \mathrm{E}T_n$ is denoted by ρ and it is called the ***traffic intensity***. For the *M/M/*1 queue $\{a,b\}$ the traffic intensity is $\rho = b/a$. We warn the reader that we shall denote with the same letter both the distribution and its distribution function. If A is a Borel set, $F(A)$ means its probability. But if x is a real, then $F(x)$ means actually $F((-\infty, x])$. If we keep that in mind, there is no danger of confusion.

Le us come back to our question: if we have two *GI/GI/*1 queues, denoted by $\{F,G\}$ (or $\{S,T\}$) and $\{F',G'\}$ (or $\{S',T'\}$) , which is better?

2. STOCHASTIC DOMINATION

For two deterministic queues, $[(S_n)_n\ ;\ (t_n)_n]$ and $[(S'_n)_n\ ;\ (t'_n)_n]$ it was easy to say that the first is better than the second if the waiting time W_n for the nth customer, is not greater than the same waiting time W'_n for the second one, for any $n \geq 1$.

If we deal with two *GI/GI/*1 queues, $\{F,G\}$ and $\{F',G'\}$, then W_n and W'_n are random variables, possible independent (for instance when the four sequences of i.i.d. random variables (S_n), (T_n), (S'_n), (T'_n) are independent. But if X and Y are independent, we cannot expect any relation such as $X \leq Y$ to exist between them. Nevertheless, if we know their distributions F_X and F_Y we can say that $X \prec_{\mathbf{st}} Y$ (X is stochastically dominated by Y) if there exist some probability space $(\Omega', \mathcal{K}', P')$ and some other random variables on it, let's say X' and Y' **with the same distributions** as X and Y , such that $X' \leq Y'$.

It is well known (see, for instance [6,7,8]) that $X \prec_{\mathbf{st}} Y \Leftrightarrow F_X(x) \geq F_Y(x)\ \forall\ x \in \Re$.

Definition. Let $\{S,T\}$ and $\{S',T'\}$ be two *GI/GI/*1 *FCFS* queues. We say that $\{S,T\}$ is **better** than $\{S',T'\}$ – and denote that by $\{S,T\} \prec \{S',T'\}$ – if $W_n \prec_{\mathbf{st}} W'_n\ \forall\ n \geq 1$.

Or, explicitely, let $(S_n)_n$, $(T_n)_n$, $(S'_n)_n$, $(T'_n)_n$ be four independent sequences of i.i.d. random variables. Let $W_1 = W'_1 = 0$ and , for $n \geq 1$, $W_{n+1} = (W_n + S_n - T_n)_+$, $W'_{n+1} = (W'_n + S'_n - T'_n)_+$. Then

$$\{S,T\} \prec \{S',T'\} \Leftrightarrow \qquad W_n \prec_{\mathbf{st}} W'_n\ \forall\ n. \tag{2.1}$$

We see that what really matter are **not** S_n and T_n, but their differences $\xi_n = S_n - T_n$ .The following result is pretty obvious (and well known – see [7])

Proposition 2.1. Let $(S_n)_n$, $(T_n)_n$, $(S'_n)_n$, $(T'_n)_n$ be four independent sequences of positive i.i.d. random variables. Let $\xi_n = S_n - T_n$ and $\xi'_n = S'_n - T'_n$. Suppose that $\xi_n \prec_{\mathbf{st}} \xi'_n$. Then $\{S,T\} \prec \{S',T'\}$.

Proof. Induction. Suppose that $W_n \prec_{\mathbf{st}} W'_n$. Then $W_n + \xi_n \prec_{\mathbf{st}} W_n + \xi'_n \prec_{\mathbf{st}} W'_n + \xi'_n$ (the invariance of the stochastic domination with respect to convolutions). Therefore $(W_n + \xi_n)_+ \prec_{\mathbf{st}} (W'_n + \xi'_n)_+$ (the mapping $f(x) = x_+$ is nondecreasing and we know that in this case $X \prec_{\mathbf{st}} Y \Rightarrow f(X) \prec_{\mathbf{st}} f(Y)$ – see, for instance [6, 8]). In other words, $W_{n+1} \prec_{\mathbf{st}} W'_{n+1}$. □

We simplify the things if we deal only with two sequences of i.i.d. random variables instead of four.

Definition . Let $(\xi_n)_n$ and $(\xi'_n)_n$ be two sequences of i.i.d. random variables having the distributions F *and* F'. Let also ξ and ξ' be two random variables such that $F_\xi = F$ and $F_{\xi'} = F'$. Consider the sequences $(W_n)_n$ and $(W'_n)_n$ given by the recurrence

$$W_1 = W'_1,\ n \geq 1 \Rightarrow W_{n+1} = (W_n + \xi_n)_+,\ W'_{n+1} = (W'_n + \xi'_n)_+. \tag{2.2}$$

Then we say that ξ is better than ξ' (and write $\xi \prec_{++} \xi'$) if $W_n \prec_{\mathbf{st}} W'_n\ \forall\ n \geq 1$.

In these terms we can write

Proposition 2.2. Let $\{S,T\}$ and $\{S',T'\}$ be two GI/GI/1 FCFS queues. Then $\{S,T\}$ is better than $\{S',T'\}$ iff $\xi \prec_{++} \xi'$, where $\xi = S - T$ and $\xi' = S' - T'$. Moreover, $\xi \prec_{\mathbf{st}} \xi' \Rightarrow \xi \prec_{++} \xi'$.

We intend to compare these two types of stochastic domination, "$\prec_{st}$" and "$\prec_{++}$".

Now we restate the same result in terms of distributions.

Suppose that W and ξ are two independent random variables with distributions G and F. The distribution of $W + \xi$ is, of course, $G * F$. What is the distribution of $(W + \xi)_+$? Let us denote it by $G \bullet F$. Notice that in our case W stands for W_n – and, in that case it is non-negative – and ξ stands for ξ_n. Formally, we can write

$$(G \bullet F)\,(t) = 0 \text{ if } t < 0 \text{ and } (G \bullet F)\,(t) = (G * F)\,(t) \text{ if } t \geq 0 \tag{2.3}$$

in terms of distribution functions. Or, if terms of distributions

$$G \bullet F = (G * F) \circ f^{-1} = (G * F)(0\text{-})\delta_0 + (1 - (G * F)(0\text{-}))F_{|[0,\infty)} \tag{2.4}$$

with $f(x) = x_+$ and $F_{|B}(A) := F(A \cap B) / F(B)$ the distribution F conditioned by the Borel set B. Here δ_0 is the Dirac distribution concentrated at 0, $\delta_0(A) = 1_A(0)$.

If we accept the notation $F_{(+)}$ instead of $F \circ f^{-1}$ – which seems to be suggestive – we can write

$$G \bullet F = (G * F)_{(+)}. \tag{2.5}$$

Let us keep in mind that

$$F_{(+)} = p\delta_0 + qF_{|[0,\infty)} \text{ with } p = F(0\text{-}) = F((-\infty,0)),\ q = 1 - p \tag{2.6}$$

for any distribution F.

Example 1. If $F = \sum_{n=-\infty}^{\infty} p_n \varepsilon_n$ then $F_{(+)} = (\sum_{n=-\infty}^{0} p_n)\varepsilon_0 + \sum_{n=1}^{\infty} p_n \varepsilon_n$.

Example 2. Suppose that a, $b > 0$, $\xi = S - T$, S,T are independent, $S \sim$ Exponential(a) and $T \sim$ Exponential(b). Write in short E_a, E_b instead of Exponential(a), Exponential(b). The density of E_a will be denoted by $e_a(x) = ae^{-ax}1_{[0,\infty)}(x)$; the distribution of $-T$ will be denoted by E_b and its density by e_b . We say that this is a negative exponential distribution. Its distribution function and density are

$$E\_b(x) = e^{bx}1_{(-\infty,0)}(x) + 1_{[0,\infty)}(x),\ e\_b(x) = be^{bx}1_{(-\infty,0)}(x). \tag{2.7}$$

In this case the distribution of ξ will be $P\circ\xi^{-1} = E_a * E\_{b}$. This distribution will be denoted by $F_{a,b}$ and its density by $f_{a,b}$. The reader can easily check that

$$f_{a,b}(x)=\frac{ab}{a+b}\left(e^{bx}1_{(-\infty,0)}(x)+e^{-ax}1_{[0,\infty)}(x)\right)\Leftrightarrow f_{a,b}=\frac{a}{a+b}e_{-b}+\frac{b}{a+b}e_a . \tag{2.8}$$

Equivalently

$$F_{a,b} = E_a * E\_{b} = pE\_{b} + qE_a, \text{ with } p=\frac{a}{a+b}=\frac{ET}{E(S+T)},\ q=\frac{b}{a+b}=\frac{ES}{E(S+T)} . \tag{2.9}$$

Now , (2.9) easily implies that

$$E_a\bullet E\_{b} = (F_{a,b})_+ = p\varepsilon_0 + qE_a \tag{2.10}$$

Remark. The difference of two independent random variables exponentially distributed is thus a mixture of an exponential and a negative exponential. The same property holds for the discrete analog of the exponential –negative binomial distributions.

We believe that the only absolute continuous distributions F,G on $[0,\infty)$ with the property that F $*$ G_ = pG_+ (1 – p)F for some $0 \le p \le 1$ are the exponential ones.

Theorem 2.3. Consider the distributions Fa,b from Example 2. Then

$$F_{a,b} \prec_{\mathbf{st}} F_{a',b'} \Leftrightarrow a \ge a',\ b \le b' \Leftrightarrow E_a \prec_{\mathbf{st}} E_{a'},\ E_{b'} \prec_{\mathbf{st}} E_b \tag{2.11}$$

$$F_{a,b} \prec_{++} F_{a',b'} \Leftrightarrow a \ge a',\ b/a \le b'/a' \tag{2.12}$$

Remark. Let $\{S,T\}$ and $\{S',T'\}$ be two *GI/GI/*1 *FCFS* queues. Let $S \sim E_a$, $S' \sim E_{a'}$, $T \sim E_b$ and $T' \sim E_{b'}$. So, (2.11) says that if $S \prec_{\mathbf{st}} S'$ (i.e. the service time is smaller in the first queue) and $T' \prec_{\mathbf{st}} T$ (i.e. the interarrival time is greater) then $\xi \prec_{\mathbf{st}} \xi'$ hence, by Proposition 2.1, the first queue is better.

Recall that the ratio $\rho = b / a = ES / ET$ is the traffic intensity of the queue $\{S.T\}$. Then (2.12) says that $\{S,T\}$ is better than $\{S',T'\}$ if and only if both the service time and the intensity of traffic are smaller in the first queue.

Proof. (**The easy part).** This is (2.11). From (2.10) we have

$$F_{a,b}(t)=\begin{cases} pe^{bt} & if \quad t<0 \\ 1-qe^{-at} & if\ t\ge 0\end{cases},\ F_{a',b'}(t)=\begin{cases} p'e^{b't} & if \quad t<0 \\ 1-q'e^{-a't} & if\ t\ge 0\end{cases}$$

with $p' = \frac{a'}{a'+b'}=\frac{ET'}{E(S'+T')}$, $q' = \frac{b'}{a'+b'}=\frac{ES'}{E(S'+T')}$. Since $F_{a,b} \prec_{\mathbf{st}} F_{a',b'} \Leftrightarrow F_{a,b}(t) \ge F_{a',b'}(t)\ \forall\ t$ we see that, if $t < 0$, that implies the inequality $pe^{bt} \ge p'e^{b't}\ \forall\ t < 0 \Leftrightarrow pe^{-bt} \ge p'e^{-b't}\ \forall\ t > 0 \Leftrightarrow e^{(b'-b)t} \ge p'/p\ \forall\ t > 0$. Letting $t \to \infty$ we get $b' \ge b$. For $t > 0$ we have the inequality $1 - qe^{-at} \ge 1 - q'e^{-a't} \Leftrightarrow q/q' \le e^{(a-a')t}\ \forall\ t > 0$ hence $a \ge a'$. Conversely, if $a \ge a'$ and $b \le b'$ then $S \prec_{\mathbf{st}} S'$ and $T' \prec_{\mathbf{st}} T$ hence $\xi \prec_{\mathbf{st}} \xi' \Leftrightarrow F_{a,b} \prec_{\mathbf{st}} F_{a',b'}$.

The difficult part. The easy part of it is "$\Rightarrow$".

Suppose that $F_{a,b} \prec_{++} F_{a',b'}$. This means that $W_n \prec_{\mathrm{st}} W'_n\ \forall\ n \ge 1$, where W_n and W'_n are constructed by the recurrence (2.2). Let G_n and G'_n be the distributions of W_{n+1} and W'_{n+1}. Then

$$G_0 = G'_0 = \varepsilon_0,\ G_{n+1} = (G_n * F_{a,b})_{(+)},\ G'_{n+1} = (G'_n * F_{a',b'})_{(+)} \tag{2.13}$$

Our assumption is that $G_n \prec_{\mathbf{st}} G'_n\ \forall\ n \ge 1$. This means that $G_1 \prec_{\mathbf{st}} G'_1 \Leftrightarrow (F_{a,b})_+ \prec_{\mathbf{st}} (F_{a',b'})_+$, hence means that $p\varepsilon_0 + qE_a \prec_{\mathbf{st}} p'\varepsilon_0 + q'E_a$ (by 2.10) $\Leftrightarrow 1 - qe^{-at} \ge 1 - q'e^{-a't} \Leftrightarrow q/q' \le e^{(a-a')t}\ \forall\ t > 0 \Leftrightarrow a \ge a'$

and $q \le q'$. But $q \le q' \Leftrightarrow \frac{b'}{a'+b'} \ge \frac{b}{a+b} \Leftrightarrow \frac{a'+b'}{b'} \le \frac{a+b}{b} \Leftrightarrow a'/b' \le a/b \Leftrightarrow b/a \le b'/a'$ or, in other words, $\rho \le \rho'$.

The difficult part is the opposite implication **"⇐".** Now, we know that $a \ge a'$ and $\rho \le \rho'$ and have to prove that $G_n \prec_{\mathbf{st}} G'_n$.

Notice that $\rho \le \rho'$ amounts to $p \ge p'$. We shall work with the distributions $G_n(a,p)$. First, we establish a more explicit recurrence for G_n that will help. We have $G_0 = \varepsilon_0$, $G_1 = p\varepsilon_0 + qE_a$. Let us compute G_2:

$$\begin{aligned} G_2 &= (G_2 * F_{a,b})_{(+)} = [(p\varepsilon_0 + qE_a)*(pE\_{b} + qE_a)]_{(+)} \text{ (by (2.9))}\\ &= (p^2\varepsilon_0 + pqE_a + pqE_a*E\_{b} + q^2E_a*E_a)_{(+)}\\ &= [p^2\varepsilon_0 + pqE_a + pq(pE\_{b} + qE_a) + q^2E_a*E_a]_{(+)} \text{ (again by (2.9)}\\ &= [p^2\varepsilon_0 + p^2qE\_{b} + (pq + pq^2)E_a + q^2E_a*E_a]_{(+)}\\ &= (p^2 + p^2q)\varepsilon_0 + (pq + pq^2)E_a + q^2E_a*E_a \end{aligned}$$

If we denote by $\Gamma_n = \Gamma_n(a)$ the Erlang distribution Gamma$(n,a) = E_a^{*n}$, with the convention that $E_a^{*0} = \delta_0$, we see that there exists a pattern

$$G_0 = \Gamma_0,\ G_1 = p\Gamma_0 + q\Gamma_1,\ G_2 = (p^2 + p^2q)\Gamma_0 + (pq + pq^2)\Gamma_1 + q^2\Gamma_2 ,\ldots.$$

It seems that all the distributions G_n are mixtures of Γ_n. Indeed, if

$$G_n = \alpha_{n,0}\Gamma_0 + \alpha_{n.1}\Gamma_1 + \ldots + \alpha_{n.n}\Gamma_n \tag{2.14}$$

then

$$G_n * F_{a,b} = (\alpha_{n,0}\Gamma_0 + \alpha_{n.1}\Gamma_1 + \ldots + \alpha_{n.n}\Gamma_n)*(pE\_{b} + qE_a) = \sum_{k=0}^{n} p\alpha_{n,k}\Gamma_k * E\_{b} + \sum_{k=0}^{n} q\alpha_{n,k}\Gamma_k * E_a .$$

But $\Gamma_k*E_a = \Gamma_{k+1}$ and a short induction points out that

$$\Gamma_k*E\_{b} = p^kE\_{b} + p^{k-1}q\Gamma_1 + p^{k-2}q\Gamma_2 + \ldots + pq\,\Gamma_{k-1} + q\Gamma_k \tag{2.15}$$

Consequently ,

$$\begin{aligned} G_n * F_{a,b} = {} & (p\, a_{n,0} + p^2 a_{n,1} + p^3a_{n,2} + p^4a_{n,3} + \ldots + p^{n+1}a_{n,n})\, E\_{b}\\ & + (qa_{n.0} + pqa_{n,1} + p^2q\, a_{n,2} + p^3q\, a_{n,3} + \ldots + p^nq\, a_{n,n})\, \Gamma_1\\ & + (qa_{n,1} + pqa_{n,2} + p^2q\, a_{n,3} + \ldots + p^{n-1}q\, a_{n,n})\Gamma_2\\ & + (qa_{n,2} + pqa_{n,3} + \ldots + p^{n-2}\, a_{n,n}\ )\, \Gamma_3\\ & \ldots\ldots\ldots\ldots\ldots\ldots\\ & + (qa_{n,n-2} + pq\, a_{n,n-1} + p^2q\, a_{n,n})\, \Gamma_{n-1} + (qa_{n,n-1}a_{n,n-1} + pqa_{n,n})\Gamma_n + qa_{n,n}\Gamma_{n+1} \end{aligned}$$

Therefore,

$$G_{n+1} = (G_n * F_{a,b})_{(+)} = a_{n+1,0}\Gamma_0 + a_{n+1,1}\Gamma_1 + \ldots + a_{n+1,n}\Gamma_n + a_{n+1,n+1}\Gamma_{n+1} \tag{2.16}$$

where

$$\begin{pmatrix} a_{n+1,0}\\ a_{n+1,1}\\ a_{n+1,2}\\ \ldots.\\ a_{n+1,n-1}\\ a_{n+1,n}\\ a_{n+1,n+1} \end{pmatrix} = \begin{pmatrix} p & p^2 & p^3 & \ldots & p^{n-1} & p^n & p^{n+1}\\ q & qp & qp^2 & \ldots & qp^{n-2} & qp^{n-1} & qp^n\\ 0 & q & qp & \ldots & qp^{n-3} & qp^{n-2} & qp^{n-1}\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & 0 & \ldots & q & qp & qp^2\\ 0 & 0 & 0 & \ldots & 0 & q & qp\\ 0 & 0 & 0 & \ldots & 0 & 0 & q \end{pmatrix} \cdot \begin{pmatrix} a_{n,0}\\ a_{n,1}\\ a_{n,2}\\ \ldots.\\ a_{n,n-1}\\ a_{n,n}\\ a_{n,n} \end{pmatrix}$$

Let us denote by $Q_{n+1}\ (p)$ the infinite matrix obtained by adding lines and columns of zeroes to the above $(n+2) \times (n+1)$ columns stochastic matrix; let also $\Gamma = \Gamma(a)$ be the vector $(\Gamma_n)_{n\ge 0}$ .Thus, for instance,

$$Q_1 = \begin{pmatrix} p & 0 & 0 & \dots \\ q & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \dots \end{pmatrix}, Q_2 = \begin{pmatrix} p & p^2 & 0 & \dots \\ q & qp & 0 & \dots \\ 0 & q & 0 & \dots \\ \dots \end{pmatrix} \dots$$

In that case we may write

$$G_n = G_n(a,p) = \Gamma'(a)Q_n(p)\, Q_{n-1}(p)\dots Q_2(p)\, Q_1(p)\boldsymbol{e}, \tag{2.17}$$

where Γ' stands for the transposed of Γ and $\boldsymbol{e}' = (1,0,0,\dots)$.

Now, the vector of distributions Γ has the obvious monotonicity property

$$\Gamma_0 \prec_{\mathbf{st}} \Gamma_1 \prec_{\mathbf{st}} \Gamma_2 \prec_{\mathbf{st}} \dots . \tag{2.18}$$

which implies that the distribution functions are decreasing:

$$\Gamma_0\,(t) \geq \Gamma_1(t) \geq \Gamma_2\,(t) \geq \dots . \tag{2.19}$$

We claim that $\Gamma' Q_n(p)$ has the same monotonicity property. Indeed,

$$\Gamma' Q_n(p) = (p\Gamma_0 + q\Gamma_1,\ p^2\Gamma_0 + pq\Gamma_1 + q\Gamma_2\,,\ p^3\Gamma_0 + p^2q\Gamma_1 + pq\Gamma_2 + q\Gamma_3,\dots.)$$

and we have to check that

$$(p^k\Gamma_0 + p^{k-1}q\Gamma_1 + \dots + pq\Gamma_{k-1} + q\Gamma_k)(t) \geq (p^{k+1}\Gamma_0 + p^kq\Gamma_1 + \dots + pq\Gamma_k + q\Gamma_{k+1})(t)\ \forall\, t \tag{2.20}$$

for any k. But this is true, since $i \leq k-1 \Rightarrow\ p^iq\Gamma_{k-i}\,(t) \leq p^iq\Gamma_{k-i-1}\,(t)$ and $(p^{k+1}\Gamma_0 + p^kq\Gamma_1)(t) \leq (p^{k+1}\Gamma_0 + p^kq\Gamma_0)(t)$ due to the fact that $\Gamma_1(t) \leq \Gamma_0(t)$!) $= p^k\Gamma_0$.

Recall that we claimed: that $a \geq a'$, $p \leq p' \Rightarrow G_n(a,p) \prec_{\mathbf{st}} G_n(a',p')$. Or, if we think of G_n as being distribution functions rather than distributions, we want to show that $a \geq a'$, $p \leq p' \Rightarrow G_n(a,p) \geq G_n(a',p')$.

The fact is that if $\Gamma = (\Gamma_n(t))_n$ is a decreasing sequence of distribution functions, then the function $f_n(p) = \Gamma' Q_n(p)$ from $(0,1)$ to $[0,\infty)$ is **non - decreasing** componentwise. Indeed, its kth component is

$$g(p) = p^k\alpha_0 + qp^{k-1}\alpha_1 + \dots + qp\alpha_{k-1} + q\alpha_k \tag{2.21}$$

where $\alpha_i = \Gamma_i(t) \geq 0$ for some real t . If we take into account that $q = 1 - \mathrm{p}$, g can be also written as $g(p) = p^n(\alpha_0 - \alpha_1) + p^{n-1}(\alpha_1 - \alpha_2) + \dots + p(\alpha_{k-1} - \alpha_k) + \alpha_k$, which is a polynomial with non-negative coefficients; hence it is obviously increasing since $p > 0$.

Let us put all these facts together: first, $p \geq p' \Rightarrow \Gamma(a)'Q_n(p)$ is nonincreasing and $\Gamma(a)'Q_n(p) \geq \Gamma(a)'Q_n(p')$; next, according to the monotonicity property $\Gamma(a)'Q_n(p')Q_{n-1}(p)$ is again non-increasing and $\Gamma(a)'Q_n(p')Q_{n-1}(p) \geq \Gamma(a)'Q_n(p')Q_{n-1}(p')$. Repeating the procedure we arrive at

$$\Gamma'(a)Q_n(p)\, Q_{n-1}(p)\dots Q_2(p)\, Q_1(p)\boldsymbol{e} \geq \Gamma'(a)Q_n(p')\, Q_{n-1}(p')\dots Q_2(p')\, Q_1(p')\boldsymbol{e}. \tag{2.22}$$

But it is clear that $a \geq a' \Rightarrow \Gamma(a) \geq \Gamma(a')$ componentwise hence from (2.22) we see that

$$\Gamma'(a)Q_n(p')\, Q_{n-1}(p')\dots Q_2(p')\, Q_1(p')\boldsymbol{e}\ \geq\ \Gamma'(a')Q_n(p')\, Q_{n-1}(p')\dots Q_2(p')\, Q_1(p')\boldsymbol{e}$$

which, corroborated with (2.17) proves that $G_n(a,p)(t) \geq G'_n(a',p')(t)$. Or, in terms of distributions, that $G_n(a,p) \prec_{\mathbf{st}} G_n(a',p') \Leftrightarrow W_n \prec_{\mathbf{st}} W'_n\ \forall\ n \geq 1$.

Remark. In the proof we heavily used the exceptional property of the exponential distributions that if X,Y are exponentially distributed and independent, then the distribution of $X - Y$ is a mixture of positive and negative exponentials. This result makes clear when we can decide which is the best between two *M/M/*1 queues. Intuitively, at least an implication should hold in general: if the service time is smaller in the first queue and the intensity of the traffic is smaller, too, then the first queue should be better then the second one. That would be nice. However, it is not true.

The conjecture "*If* $S \prec_{\mathbf{st}} S'$ *and* $\rho \leq \rho'$ *then the first queue is better*" is false.

Counterexample 2.4. *Let* $S \sim \begin{pmatrix} 1 & 2 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$, $T \sim \begin{pmatrix} 1 & 3 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$, $S' \sim \begin{pmatrix} 1 & 2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, $T' \sim \begin{pmatrix} 1 & 3 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, *with all these random variables independent. Let* $\xi = S - T$ *and* $\xi' = S' - T'$. *Then*

- *it is not true that* $\xi \prec_{\mathbf{st}} \xi'$, $S' \prec_{\mathbf{st}} S$ (*contrary to the above conjecture*),
- $\rho = \rho'$ and
- $\xi \prec_{++} \xi'$.

Proof. We have ES = 7/4, ET = 7/3, ES' = 3/2, ET' = 4/2 $\Rightarrow \rho = \rho' = 3/4$. Obviously, $S' \prec_{\mathbf{st}} S$. At a first glance, the queue $\{S,T\}$ should be better than $\{S',T'\}$. But,

$\xi \sim \begin{pmatrix} -2 & -1 & 0 & 1 \\ \frac{2}{12} & \frac{6}{12} & \frac{1}{12} & \frac{3}{12} \end{pmatrix} = (\alpha\varepsilon_{-2} + \beta\varepsilon_{-1} + \gamma\varepsilon_0 + \varepsilon_1)/4$ with $\alpha = 2/3$, $\beta = 2$, $\gamma = 1/3$, and $\xi' \sim \begin{pmatrix} -2 & -1 & 0 & 1 \\ \frac{3}{12} & \frac{3}{12} & \frac{3}{12} & \frac{3}{12} \end{pmatrix} = (\varepsilon_{-2} + \varepsilon_{-1} + \varepsilon_0 + \varepsilon_1)/4 = \text{Uniform}(\{-2,-1,0,1\})$.

The cumulative distribution functions are $F := F_\xi = \begin{pmatrix} -2 & -1 & 0 & 1 \\ \frac{2}{12} & \frac{8}{12} & \frac{9}{12} & \frac{12}{12} \end{pmatrix}$ and $F' := F_{\xi'} = \begin{pmatrix} -2 & -1 & 0 & 1 \\ \frac{3}{12} & \frac{6}{12} & \frac{9}{12} & \frac{12}{12} \end{pmatrix}$. As they are not comparable, there is no stochastic domination between ξ and ξ'.

Consider the waiting times W_n and W_{n+1} given by the recurrences $W_1 = W'_1 = 0$ and $n \ge 1 \Rightarrow W_{n+1} = (W_n+\xi_n)_+$, $W'_{n+1} = (W'_n+\xi'_n)_+$ with distributions G_n and G'_n. We claim that $G_n \prec_{\mathbf{st}} G'_n$ for any $n \ge 1$. We have $G_1 = G'_1 = \varepsilon_0$ and $G_2 = G'_2 = (\varepsilon_0 * F_\xi)_{(+)} = \begin{pmatrix} 0 & 1 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$. In terms of distributions, the recurrence can be written as $G_{n+1} = (G_n * F)_{(+)}$, $G'_{n+1} = (G'_n * F')_{(+)}$. To see what happens, we shall work with a bit more general distribution for F, namely

$$F = (\alpha\varepsilon_{-2} + \beta\varepsilon_{-1} + \gamma\varepsilon_0 + \varepsilon_1)/4 \text{ with } \alpha,\beta,\gamma > 0,\ \alpha + \beta + \gamma = 3. \tag{2.23}$$

We claim that $\gamma < \alpha$, $\beta \ge 2 \Rightarrow G_n \prec_{\mathbf{st}} G'_n\ \forall\, n$. Actually, we have a more general claim, namely that

$$\gamma < \alpha,\ \beta \ge 2 \Rightarrow (G_n * F)_{(+)} \prec_{\mathbf{st}} (G_n * F')_{(+)}\ \forall\, n \ge 1. \tag{2.24}$$

The reason is that if (2.24) is true could prove that $G_n \prec_{\mathbf{st}} G'_n$ by induction: if that holds for n, then $G_{n+1} = (G_n * F)_{(+)} \prec_{\mathbf{st}} (G_n * F')_{(+)} \prec_{\mathbf{st}} (G'_n * F')_{(+)}$ due to the monotonicity of the operation "$\bullet$" (obviously $G \prec_{\mathbf{st}} G'$ does imply that $G\bullet F \prec_{\mathbf{st}} G'\bullet F$). So we shall prove (2.24). For $n = 1$ it is true: $G_1\bullet F = G_1\bullet F' = \begin{pmatrix} 0 & 1 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$.

Suppose that $G_n = p_0\varepsilon_0 + p_1\varepsilon_1 + p_2\varepsilon_2 + \ldots$ .The reader is invited to check that

$$\begin{aligned} 4G_n\bullet F &= [3p_0 + (\alpha+\beta)p_1 + \alpha p_2]\delta_0 + [p_0 + \gamma p_1 + \beta p_2 + \alpha p_3]\delta_1 + [p_1 + \gamma p_2 + \beta p_3 + \alpha p_4]\delta_2 + \ldots = \\ &= [3p_0 + (\alpha+\beta)p_1 + \alpha p_2]\delta_0 + \sum_{n=1}^{\infty}(p_{n-1} + \gamma p_n + \beta p_{n+1} + \alpha p_{n+2})\delta_n \end{aligned} \tag{2.25}$$

While

$$4G_n\bullet F' = [3p_0 + 2p_1 + p_2]\delta_0 + [p_0 + p_1 + p_2 + p_3]\delta_1 + [p_1 + p_2 + p_3 + p_4]\delta_2 + \ldots \quad = \tag{2.26}$$

$$[3p_0 + 2p_1 + p_2]\delta_0 + \sum_{n=1}^{\infty}(p_{n-1} + p_n + p_{n+1} + p_{n+2})\delta_n .$$

The cummulative distribution functions are

$$(4\ G_n \bullet F)\ (n) = 4G_n(n\text{-}1) + 3p_n + (\alpha+\beta)p_{n+1} + \alpha\ p_{n+2} \quad \forall\ n \ge 1 \tag{2.27}$$

And

$$(4\ G_n \bullet F')(n) = 4G_n(n\text{-}1) + 3p_n + 2p_{n+1} + p_{n+2} \quad \forall\ n \ge 1. \tag{2.28}$$

Therefore, the condition (2.24) is equivalent to

$$(\alpha + \beta)\ p_n + \alpha p_{n+1} \ge 2p_n + p_{n+1} \Leftrightarrow (1-\gamma)\ p_n \ge (\ 1-\alpha\ )\ p_{n+1} \quad \forall\ n \ge 1. \tag{2.29}$$

We shall check by induction that G_n have this property if $\beta \ge 2$. Actually, G_n have a stronger property, namely that

$$(1-\gamma)\ p_n \ge (\ 1-\alpha\ )\ p_{n+1} \quad \forall\ n \ge 0. \tag{2.30}$$

Suppose that (2.30) holds for G_n. Let us write $4G_n \bullet F = \pi_0\varepsilon_0 + \pi_1\varepsilon_1 + \pi_2\varepsilon_2 + \ldots$.. with

$$\pi_n = p_{n-1} + \gamma p_n + \beta p_{n+1} + \alpha p_{n+2} \text{ if } n \ge 1,\ \pi_0 = 3p_0 + (\alpha+\beta)p_1 + \alpha p_2 . \tag{2.31}$$

Assume that (2.30) holds. For $n \ge 1$ we see that

$$(1-\gamma)\pi_n + (\alpha-1)\pi_{n+1} = [(1-\gamma)p_{n-1} + (\alpha-1)p_n] + \gamma\ [(1-\gamma)p_n + (\alpha-1)p_{n+1}] + \beta[(1-\gamma)p_{n+1} + (\alpha-1)p_{n+2}]$$
$$+\alpha[(1-\gamma)p_{n+2} + (\alpha-1)p_{n+3}]$$

hence is nonnegative due to our induction hypothesis. The only problem is to check that

$$\alpha + \gamma \le 1 \ \Rightarrow (1-\gamma)\pi_0 \ge (1 - \alpha)\ \pi_1 \quad \Leftrightarrow \pi_0 \ge \frac{1-\alpha}{1-\gamma}\pi_1 . \tag{2.32}$$

If (2.32) is true, that will complete our proof. Notice that (2.32) is equivalent to

$$3p_0 + (3 - \gamma)\ p_1 + \alpha p_2 \ge \frac{1-\alpha}{1-\gamma}(p_0 + \gamma p_1 + \beta p_2 + \alpha p_3). \tag{2.33}$$

As $\alpha p_2 \ge \frac{1-\alpha}{1-\gamma}\alpha p_3$, (2.33) will be true if $3p_0 + (3 - \gamma)\ p_1 \ \ge \frac{1-\alpha}{1-\gamma}(p_0 + \gamma p_1 + \beta p_2\ )$ or, equivalently, if $(3-\frac{1-\alpha}{1-\gamma})p_0 + (3 - \gamma)p_1 \ge \ \frac{1-\alpha}{1-\gamma}(\gamma p_1 + \beta p_2\ )$. But $\frac{1-\alpha}{1-\gamma}\gamma p_1 + \beta\frac{1-\alpha}{1-\gamma}p_2 \ \le \ \frac{1-\alpha}{1-\gamma}\gamma p_1 + \beta p_1 \le (\gamma + \beta)p_1$ (recall that $1 - \alpha \le 1 - \gamma$!) $= (3-\alpha)p_1 \le (3 - \gamma)p_1 \le 2p_0 + (3 - \gamma)p_1 \le (3-\frac{1-\alpha}{1-\gamma})p_0 + (3 - \gamma)p_1$ hence (2.33) is true. So, we have proved our claim (2.24). □

Remark. It is annoying that we could not find equivalent conditions for "$\xi \prec_{++} \xi'$ " in a general situation when $S \sim (1-\text{p})\ \varepsilon_1 + p\varepsilon_2$ and $T \sim (1-p')\varepsilon_1 + p\varepsilon_3$.

The condition for "$\xi \prec_{\mathbf{st}} \xi'$ " is easier: namely $p' \ge \max\left(\frac{1}{2}, \frac{1}{4(1-p)}\right)$).

3. COMPARING TWO QUEUES WITH CYCLING

It may happen sometimes, as studied in [1],[2],[3], [4],[5] that a customer who arrives and find the server busy goes away and returns after a time period, θ. Even if in fact this θ is a random variable itself, sometimes it is not far from truth to consider it a constant. For example, if an airplane comes and finds the track busy, it makes a cycle and then comes again; or in the computer case, if the data highway is busy, there exists a built–in delay and the same thing happens. In that case the waiting time of the nth customer is given by

$$W_1 = 0 \text{ and } n \geq 1 \Rightarrow W_{n+1} = \theta\left\lceil \left(\frac{W_n + S_n - T_n}{\theta}\right)_+ \right\rceil \tag{3.1}$$

Indeed, suppose that the $(n+1)$th customer arrives at t_{n+1}. In a normal situation his waiting time would be $(W_n + \xi_n)_+$ with $\xi_n = S_n - T_n$. If $W_n + \xi_n \leq 0$, he does not wait anymore. But, if $W_n + \xi_n > 0$, he departs and comes again after a time θ; if finds again the system busy, he departs again and so on, until the system becomes free. The number of departures is, of course, $\left\lceil \frac{W_n + \xi_n}{\theta} \right\rceil$, where $\lceil x \rceil$ is the first integer from right of x, defined by $\lceil x \rceil = k$ iff $k - 1 < x \leq k$. Denote such a queuing system by $\{S,T,\theta\}$. The condition that such a queuing system have a stationary distribution for W_n is similar to the usual one.

Proposition 3.1. *Let us consider a queue $\{S,T,\theta\}$. Let $X_n = W_n/\theta$. Then X_n satisfy the recurrence*

$$X_1 = 0 \text{ and } n \geq 1 \Rightarrow X_{n+1} = \left(X_n + \left\lceil \frac{\xi_n}{\theta} \right\rceil\right)_+ \tag{3.2}$$

with $\xi_n = S_n - T_n$, which is similar to (1.4). *Consequently, $(W_n)_n$ has a stationary distribution if and only if* $E\left\lceil \frac{\xi_n}{\theta} \right\rceil < 0$.

Proof. We write (3.1) as $X_{n+1} = \left\lceil \left(X_n + \frac{\xi_n}{\theta}\right)_+ \right\rceil = \left(\left\lceil X_n + \frac{\xi_n}{\theta} \right\rceil\right)_+ = \left(X_n + \left\lceil \frac{\xi_n}{\theta} \right\rceil\right)_+$ since always $\lceil x_+ \rceil = (\lceil x \rceil)_+$ and if k is an integer, then $\lceil k + x \rceil = k + \lceil x \rceil$. □

It means that if we want to compare two queues $\{S,T,\theta\}$ and $\{S',T',\theta'\}$, what we have to do is to compare the random variables $\left\lceil \frac{S-T}{\theta} \right\rceil$ and $\left\lceil \frac{S'-T'}{\theta'} \right\rceil$.

Remark. The fact that if $E\left\lceil \frac{\xi_n}{\theta} \right\rceil < 0$ then W_n has a limit distribution (stationary distribution) was proved in [1] using a different technique.

Remark. It is easy to see that for any random variable X we have $\lim_{\theta\to\infty} \left\lceil \frac{X}{\theta} \right\rceil = 1_{(X > 0)}$ hence $\lim_{\theta\to\infty} E\left\lceil \frac{X}{\theta} \right\rceil = P(X > 0)$. Thus, if $P(\xi_n > 0)$ is positive, there **always** exists some θ such that $\{S,T,\theta\}$ has no limit distribution for W_n. Equivalently, **if θ is too great, almost any queuing system $\{S,T\}$ becomes non feasible** since $W_n \to \infty$. The only case – hardly met in reality – when this phenomenon does not occur is when ess sup $S \leq$ ess inf T.

ACKNOWLWDGEMENTS.

The research was partly supported by the Hungarian National Foundation for Scientific Research under grant OTKA K60698/2005 and partially supported by an action of the program ECO-NET 2006 financed by the French government.

REFERENCES

1. E.B. KOBA, *On a GI/G/*1 *retrial queuing system FCFS service discipline.* Dopovidi NAN Ukrainy, **6**,pp.101 – 103 , 2000. (in Russian).
2. K. LAEVENS, B. van HOUDT., C. BLONDIA and H. BRUNEL, *On the sustanaible load of fiber delay line buffers.*Electronics Letters, **40** ,pp. 137 – 138, 2004.
3. L. LAKATOS, *On a simple continuous cycling waiting problem.* Ann. Univ. Sci. Budapest, Sect. Comp, **14**, pp. 105 – 113, 1994.
4. L. LAKATOS, *A retrial system with time limited tasks*. Theory of stochastic processes, **8** [24], pp 249 - 255, 2002.
5. K. V. MYCHALEVYCH, *A comparison of a classical retrial M/G/1 queuing system and a Lakatos – type M/G/1 cycling – waiting time queuing system.,* Ann. Univ. Sci. Budapest, Sect. Comp, **23**, pp. 229 – 238, 2004.
6. M. SHAKED and J. SHANTIKUMAR, *Stochastic Orderings and Their Applications*. Academic Press, NewYork 1993.
7. R. SZEKLI, *Stochastic Ordering and Dependence in Applied Probability.* Springer, Berlin, 1995.
8. G. ZBĂGANU, *Mathematical Methods in Risk Theory and Actuaries.* University Press*,* Bucharest 2004 (in Romanian).

Received April 25, 2007