THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 8, Number 2/2007, pp. 000-000

UNSTEADY MOTIONS OF A MAXWELL FLUID DUE TO LONGITUDINAL AND TORSIONAL OSCILLATIONS OF AN INFINITE CIRCULAR CYLINDER

Constantin FETECĂU, Corina FETECĂU

Technical University of Iaşi, România
E-mail: cfetecau@yahoo.de

The exact solutions for the motions of a Maxwell fluid due to longitudinal and torsional oscillations of an infinite circular cylinder are determined. These solutions, presented as sum of the steady-state and transient solutions, reduce to those for a Newtonian fluid as a limiting case. The steady-state solutions are also obtained for large values of time *t*.

Key words: Maxwell fluid; Longitudinal and torsional oscillations; Starting solutions.

1. INTRODUCTION

Flows in the neighborhood of spinning or oscillating bodies are of interest to both academic workers and industry. Among them, the flows between oscillating cylinders are some of the most important and interesting problems of motion near oscillating bodies. As early as 1886, Stokes [1] established an exact solution for the rotational oscillations of an infinite rod immersed in a classical linearly viscous fluid. Casarella and Laura [2] obtained an exact solution for the problem of the rod undergoing both torsional and longitudinal oscillations in a Newtonian fluid. Later, Rajagopal [3] presents two simple but elegant solutions for the flow of a second grade fluid induced by the longitudinal and torsional oscillations of an infinite rod. Their solutions have been recently extended to Oldroyd-B fluids by Rajagopal and Bhatnagar [4]. However, we want to point out that all previous solutions are steady-state solutions, while in order to obtain a starting solution, describing the flow at small and large times after the start of the boundary wall, a transient solution has to be added to the steady-state solution.

The aim of this paper is to study the motion of a Maxwell fluid due to the longitudinal and torsional oscillations of an infinite circular cylinder. Actually, we establish the starting solutions corresponding to such flows between infinite concentric circular cylinders and through a circular cylinder. Starting solutions for the motion of a non-Newtonian fluid due to an oscillating wall have been recently established in [5, 6]. These solutions, depending of the initial conditions, are presented as sum of the steady-state and transient solutions. For large times they tend to the steady-state solutions which are independent of initial conditions and periodic in time. Following Rajagopal [3], the steady-state solutions corresponding to the mentioned problems are also presented in simpler forms, in terms of the modified Bessel functions. In the special case, when the relaxation time $\lambda \to 0$, all solutions are going to those for a Newtonian fluid.

2. GOVERNING EQUATIONS

The Cauchy stress tensor $\mathbf{T}$ in an incompressible Maxwell fluid is given by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^{\mathrm{T}}) = \mu\mathbf{A}, \tag{1}$$

where $-p\mathbf{I}$ is the indeterminate part of the stress due to the constraint of incompressibility, $\mathbf{S}$ the extra-stress tensor, $\mathbf{A}$ the first Rivlin-Ericcksen tensor, $\mathbf{L}$ the velocity gradient, μ the dynamic viscosity, λ the relaxation time and the upper dot denotes material time differentiation.

The model (1) is consistent with some important microscopically models of polymers and its predictions of the normal-stress differences are qualitatively acceptable. It has been quite useful in the study of dilute polymeric fluids in viscoelasticity.

In the following we shall seek a velocity field of the form [3, 4]

$$\mathbf{v} = \mathbf{v}(r,t) = w(r,t)\mathbf{e}_\theta + \mathrm{v}(r,t)\mathbf{e}_z, \tag{2}$$

where $\mathbf{e}_\theta$ and $\mathbf{e}_z$ denote the unit vectors along the θ and z directions of the cylindrical coordinate system r, θ and z. For such flows, the constraint of incompressibility is automatically satisfied. Since the velocity field $\mathbf{v}$ is independent of θ and z, we are expecting that the extra-stress tensor $\mathbf{S}$ to be also a function of r and t only. Further, due to the rotational symmetry $\partial_\theta p = 0$ [4].

On substituting (1) and (2) into the balance of linear momentum, neglecting the body forces and assuming that there is no applied pressure gradient along the axial direction, one attains to the linear partial differential equations [7] (see also [4], Eqs. (28) and (35) for $\lambda_2 = 0$)

$$\lambda\partial_t^2 w(r,t) + \partial_t w(r,t) = \nu\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)w(r,t), \tag{3}$$

$$\lambda\partial_t^2 \mathrm{v}(r,t) + \partial_t \mathrm{v}(r,t) = \nu\left(\partial_r^2 + \frac{1}{r}\partial_r\right)\mathrm{v}(r,t), \tag{4}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid and ρ its constant density.

The uncoupled equations (3) and (4) with appropriate boundary and initial conditions can be solved in general by several methods. The Laplace transform can be used to eliminate the time variable. However, the inversion procedure for obtaining the solution is not always a trivial mater. Further, the solution so obtained for a second grade fluid does not satisfy the initial condition [8]. This is due to the incompatibility between the prescribed data. Here, we shall use the finite Hankel transforms. It is worthwhile pointing out that, in general, the governing equations for a Maxwell fluid are one order higher in time than the corresponding equations for a Newtonian fluid. Consequently, in order to solve a well-posed problem for such a fluid one has to require additional initial conditions [7].

3. MOTION BETWEEN CIRCULAR CYLINDERS

Consider a Maxwell fluid at rest in an annular region between two infinitely long coaxial circular cylinders of radii R_0 and $R\,(> R_0)$. At time $t = 0^+$ the outer cylinder starts to oscillate according to [3, 4]

$$\mathbf{v} = \mathbf{v}(R,t) = W\cos(\omega_1 t)\mathbf{e}_\theta + V\cos(\omega_0 t)\mathbf{e}_z, \tag{5}$$

where ω_1 and ω_0 are the frequencies of the velocity of the cylinder. Due to the shear the fluid between cylinders is gradually moved. Its velocity is of the form (2) and the governing equations are (3) and (4). The associate initial and boundary conditions are

$$w(R_0,t) = \mathrm{v}(R_0,t) = 0 \quad \text{for all } t, \tag{6}$$

$$w(R,t) = W\cos(\omega_1 t); \quad \mathrm{v}(R,t) = V\cos(\omega_0 t) \quad \text{for all } t > 0 \tag{7}$$

and

$$w(r,0) = \mathrm{v}(r,0) = \partial_t w(r,0) = \partial_t \mathrm{v}(r,0) = 0; \quad r \in [R_0, R). \tag{8}$$

Making the change of unknown functions

$$w(R,t) = U_1(r)\cos(\omega_1 t) + u_1(r,t) \quad \text{and} \quad \mathrm{v}(R,t) = U_0(r)\cos(\omega_0 t) + u_0(r,t), \tag{9}$$

where $U_1(r) = W\dfrac{R(r^2 - R_0^2)}{r(R^2 - R_0^2)}$ and $U_0(r) = V\dfrac{\ln(r/R_0)}{\ln(R/R_0)}$, we attain to the next problems with initial and boundary conditions

$$\lambda\partial_t^2 u_n(r,t) + \partial_t u_n(r,t) = \nu\Delta_n u_n(r,t) + \omega_n U_n(r)[\sin(\omega_n t) + \lambda\omega_n \cos(\omega_n t)];\quad r \in (R_0, R),\quad t > 0, \tag{10}$$

$$u_n(R_0,t) = u_n(R,t) = 0;\quad t > 0 \tag{11}$$

and

$$u_n(r,0) = -U_n(r),\quad \partial_t u_n(r,0) = 0;\quad r \in [R_0, R), \tag{12}$$

in which $n = 0,1$ and the differential operators $\Delta_n = \partial_r^2 + \dfrac{1}{r}\partial_r - \dfrac{n}{r^2}$.

In order to obtain analytical solutions for these problems we shall use, as in [7], the well-known expansion theorem of Steklov. In view of this theorem the functions $u_n(r,t)$, whose partial derivatives $\partial_r u_n$ and $\partial_r^2 u_n$ have to be piecewise continuous for each $t > 0$, can be written as Fourier-Bessel series absolutely and uniformly convergent in terms of the eigenfunctions (see [9], Sec. 97)

$$\mathrm{B}_n(rr_{nm}) = A_{nm}\left[\mathrm{J}_n(rr_{nm}) - \frac{\mathrm{J}_n(R_0 r_{nm})}{\mathrm{Y}_n(R_0 r_{nm})}\mathrm{Y}_n(rr_{nm})\right], \tag{13}$$

of the eigenvalue problems $\Delta_n\Phi(r) + \gamma^2\Phi(r) = 0$, $\Phi(R_0) = \Phi(R) = 0$, i.e.,

$$u_n(r,t) = \sum_{m=1}^{\infty} u_{nm}(t)\mathrm{B}_n(rr_{nm}). \tag{14}$$

In the above relations $\mathrm{J}_n(\cdot)$ and $\mathrm{Y}_n(\cdot)$ denote Bessel functions in standard notations, r_{nm} are positive roots of transcendental equations $\mathrm{B}_n(Rr) = 0$ while the constants A_{nm} are chosen such that the normalization conditions

$$\int_{R_0}^{R} r\left[\mathrm{B}_n(rr_{nm})\right]^2 \mathrm{d}r = 1;\quad n = 0,1, \tag{15}$$

to be satisfied.

Now, introducing (14) into (10), multiplying then by $r\mathrm{B}_n(rr_{np})$, integrating the result with respect to r from R_0 to R and having the initial and boundary conditions (11) and (12) in mind, we find that (see [9], Sec. 98)

$$\lambda\ddot{u}_{nm}(t) + \dot{u}_{nm}(t) + \nu r_{nm}^2 u_{nm}(t) = \omega_n U_{nm}[\sin(\omega_n t) + \lambda\omega_n \cos(\omega_n t)];\quad t > 0 \tag{16}$$

and

$$u_{nm}(0) = -U_{nm},\quad \dot{u}_{nm}(0) = 0, \tag{17}$$

where U_{nm} are the modified finite Hankel transforms of $U_n(r)$.

Solving the ordinary differential equations (16) with the initial conditions (17) and taking into account Eqs. (9) and (14), we get for the starting solutions $w(r,t)$ and $\mathrm{v}(r,t)$ the expressions:

$$w(r,t) = W\cos(\omega_1 t)\frac{R(r^2 - R_0^2)}{r(R^2 - R_0^2)} + \omega_1\sum_{n=1}^{\infty}\left[\omega_1 A_{1n}\cos(\omega_1 t) + \nu B_{1n}\sin(\omega_1 t)\right]U_{1n}\mathrm{B}_1(r r_{1n}) - $$
$$- \exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty}U_{1n}\Phi_{1n}(t)\mathrm{B}_1(r r_{1n}) \tag{18}$$

and

$$\mathrm{v}(r,t) = V\cos(\omega_0 t)\frac{\ln(r/R_0)}{\ln(R/R_0)} + \omega_0\sum_{n=1}^{\infty}\left[\omega_0 A_{0n}\cos(\omega_0 t) + \nu B_{0n}\sin(\omega_0 t)\right]U_{0n}\mathrm{B}_0(r r_{0n}) - $$
$$- \exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty}U_{0n}\Phi_{0n}(t)\mathrm{B}_0(r r_{0n}), \tag{19}$$

where

$$A_{nm} = \frac{\lambda(\nu r_{nm}^2 - \lambda\omega_n^2) - 1}{(\nu r_{nm}^2 - \lambda\omega_n^2)^2 + \omega_n^2}, \quad B_{nm} = \frac{r_{nm}^2}{(\nu r_{nm}^2 - \lambda\omega_n^2)^2 + \omega_n^2},$$

$$\Phi_{nm}(t) = \begin{cases} \left[\mathrm{ch}\left(\dfrac{\alpha_{nm}t}{2\lambda}\right) + \dfrac{1}{\alpha_{nm}}\mathrm{sh}\left(\dfrac{\alpha_{nm}t}{2\lambda}\right)\right]\left(\omega_n^2 A_{nm} + 1\right) + 2\nu\lambda\omega_n^2 B_{nm}\mathrm{sh}\left(\dfrac{\alpha_{nm}t}{2\lambda}\right) & r_{nm} < \dfrac{1}{2\sqrt{\nu\lambda}} \\[2ex] \dfrac{\omega_n^2 A_{nm} + 1 + 2\nu\lambda\omega_n^2 B_{nm}}{2\lambda}t + \omega_n^2 A_{nm} + 1 & r_{nm} = \dfrac{1}{2\sqrt{\nu\lambda}} \\[2ex] \left[\cos\left(\dfrac{\beta_{nm}t}{2\lambda}\right) + \dfrac{1}{\beta_{nm}}\sin\left(\dfrac{\beta_{nm}t}{2\lambda}\right)\right]\left(\omega_n^2 A_{nm} + 1\right) + 2\nu\lambda\omega_n^2 B_{nm}\sin\left(\dfrac{\beta_{nm}t}{2\lambda}\right) & r_{nm} > \dfrac{1}{2\sqrt{\nu\lambda}}, \end{cases}$$

$$\alpha_{nm} = \sqrt{1 - 4\nu\lambda r_{nm}^2} \quad \text{and} \quad \beta_{nm} = \sqrt{4\nu\lambda r_{nm}^2 - 1}.$$

For large values of t, these solutions reduce to the steady-state solutions

$$w_{st}(r,t) = W\cos(\omega_1 t)\frac{R(r^2 - R_0^2)}{r(R^2 - R_0^2)} + \omega_1\sum_{n=1}^{\infty}\left[\omega_1 A_{1n}\cos(\omega_1 t) + \nu B_{1n}\sin(\omega_1 t)\right]U_{1n}\mathrm{B}_1(r r_{1n}), \tag{20}$$

respectively,

$$\mathrm{v}_{st}(r,t) = V\cos(\omega_0 t)\frac{\ln(r/R_0)}{\ln(R/R_0)} + \omega_0\sum_{n=1}^{\infty}\left[\omega_0 A_{0n}\cos(\omega_0 t) + \nu B_{0n}\sin(\omega_0 t)\right]U_{0n}\mathrm{B}_0(r r_{1n}), \tag{21}$$

which are periodic in time and independent of the initial conditions. However, they satisfy the governing equations (3) and (4) as well as the boundary conditions (6) and (7).

Finally, following the same way as in [3] and [4], we can find simpler forms for the steady-state solutions (20) and (21). These are

$$w_{st}(r,t) = W\,\mathrm{Re}\left\{\frac{\mathrm{I}_1(r\sqrt{\gamma_1})\mathrm{K}_1(R_0\sqrt{\gamma_1}) - \mathrm{I}_1(R_0\sqrt{\gamma_1})\mathrm{K}_1(r\sqrt{\gamma_1})}{\mathrm{I}_1(R\sqrt{\gamma_1})\mathrm{K}_1(R_0\sqrt{\gamma_1}) - \mathrm{I}_1(R_0\sqrt{\gamma_1})\mathrm{K}_1(R\sqrt{\gamma_1})}\ \mathrm{e}^{\mathrm{i}\omega_1 t}\right\} \tag{22}$$

and

$$\mathrm{v}_{st}(r,t) = V\,\mathrm{Re}\left\{\frac{\mathrm{I}_0(r\sqrt{\gamma_0})\mathrm{K}_0(R_0\sqrt{\gamma_0}) - \mathrm{I}_0(R_0\sqrt{\gamma_0})\mathrm{K}_0(r\sqrt{\gamma_0})}{\mathrm{I}_0(R\sqrt{\gamma_0})\mathrm{K}_0(R_0\sqrt{\gamma_0}) - \mathrm{I}_0(R_0\sqrt{\gamma_0})\mathrm{K}_0(R\sqrt{\gamma_0})}\ \mathrm{e}^{\mathrm{i}\omega_0 t}\right\}, \tag{23}$$

where Re denotes the real part of the complex number which follows, $\mathrm{I}_n(\cdot)$ and $\mathrm{K}_n(\cdot)$ are modified Bessel functions, $\gamma_n = \omega_n(i - \lambda\omega_n)/\nu$ and $i = \sqrt{-1}$.

4. FLOW WITHIN AN INFINITE CIRCULAR CYLINDER

Taking the limit of Eqs. (13) as $R_0 \to 0$ and having the normalization conditions (15) in mind, we obtain the eigenfunctions $\sqrt{2}\mathrm{J}_1(rr_{1n})/[R\mathrm{J}_2(Rr_{1n})]$ and $\sqrt{2}\mathrm{J}_0(rr_{0n})/[R\mathrm{J}_1(Rr_{0n})]$ corresponding to the similar flow within an infinite circular cylinder. Furthermore, the associated velocity fields

$$\begin{aligned} w(r,t) &= \frac{r}{R}W\cos(\omega_1 t) + \frac{2}{R}\omega_1^2 W\cos(\omega_1 t)\sum_{n=1}^{\infty} A_{1n}\frac{\mathrm{J}_1(rr_{1n})}{r_{1n}\mathrm{J}_2(Rr_{1n})} + \\ &+ \frac{2}{R}\nu\omega_1 W\sin(\omega_1 t)\sum_{n=1}^{\infty} B_{1n}\frac{\mathrm{J}_1(rr_{1n})}{r_{1n}\mathrm{J}_2(Rr_{1n})} - \frac{2}{R}W\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty}\Phi_{1n}(t)\frac{\mathrm{J}_1(rr_{1n})}{r_{1n}\mathrm{J}_2(Rr_{1n})} \end{aligned} \tag{24}$$

and

$$\begin{aligned} \mathrm{v}(r,t) &= V\cos(\omega_0 t) + \frac{2}{R}\omega_0^2 V\cos(\omega_0 t)\sum_{n=1}^{\infty} A_{0n}\frac{\mathrm{J}_0(rr_{0n})}{r_{0n}\mathrm{J}_1(Rr_{0n})} + \\ &+ \frac{2}{R}\nu\omega_0 V\sin(\omega_0 t)\sum_{n=1}^{\infty} B_{0n}\frac{\mathrm{J}_0(rr_{0n})}{r_{0n}\mathrm{J}_1(Rr_{0n})} - \frac{2}{R}V\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty}\Phi_{0n}(t)\frac{\mathrm{J}_0(rr_{0n})}{r_{0n}\mathrm{J}_1(Rr_{0n})}, \end{aligned} \tag{25}$$

are also obtained as limiting cases of Eqs. (18) and (19). In the above relations r_{0n} and r_{1n} are certainly the positive roots of the transcendental equations $\mathrm{J}_0(Rr) = 0$ and $\mathrm{J}_1(Rr) = 0$, respectively. Moreover, in view of [10], the entries 1 and 2 of Table X, it results that

$$\lim_{R_0\to 0}\frac{R}{R^2 - R_0^2}\int_{R_0}^{R}(r^2 - R_0^2)\mathrm{B}_1(rr_{1n})\mathrm{d}r = \frac{\sqrt{2}}{r_{1n}} \quad \text{and} \quad \lim_{R_0\to 0}\frac{1}{\ln(R/R_0)}\int_{R_0}^{R} r\ln(r/R_0)\mathrm{B}_0(rr_{0n})\mathrm{d}r = \frac{\sqrt{2}}{r_{0n}}.$$

For large times the starting solutions (24) and (25) tend to the corresponding steady-state solutions

$$w_{st}(r,t) = \frac{r}{R}W\cos(\omega_1 t) + \frac{2}{R}\omega_1^2 W\cos(\omega_1 t)\sum_{n=1}^{\infty} A_{1n}\frac{\mathrm{J}_1(r\,r_{1n})}{r_{1n}\mathrm{J}_2(R\,r_{1n})} + \frac{2}{R}\nu\omega_1 W\sin(\omega_1 t)\sum_{n=1}^{\infty} B_{1n}\frac{\mathrm{J}_1(r\,r_{1n})}{r_{1n}\mathrm{J}_2(R\,r_{1n})} \tag{26}$$

and

$$\mathrm{v}_{\mathrm{st}}(r,t) = V\cos(\omega_0 t) + \frac{2}{R}\omega_0^2 V\cos(\omega_0 t)\sum_{n=1}^{\infty} A_{0n}\frac{\mathrm{J}_0(rr_{0n})}{r_{0n}\mathrm{J}_1(Rr_{0n})} + \frac{2}{R}\nu\omega_0 V\sin(\omega_0 t)\sum_{n=1}^{\infty} B_{0n}\frac{\mathrm{J}_0(rr_{0n})}{r_{0n}\mathrm{J}_1(Rr_{0n})}. \tag{27}$$

These last two solutions can be also written in the simple forms

$$w_{st}(r,t) = W\,\mathrm{Re}\left\{\frac{\mathrm{I}_1(r\sqrt{\gamma_1})}{\mathrm{I}_1(R\sqrt{\gamma_1})}\ \mathrm{e}^{\mathrm{i}\omega_1 t}\right\},\quad \mathrm{v}_{st}(r,t) = V\,\mathrm{Re}\left\{\frac{\mathrm{I}_0(r\sqrt{\gamma_0})}{\mathrm{I}_0(R\sqrt{\gamma_0})}\ \mathrm{e}^{\mathrm{i}\omega_0 t}\right\}, \tag{28}$$

obtained again as limiting cases of (22) and (23).

5. NUMERICAL RESULTS AND CONCLUSIONS

In this paper, the velocity fields corresponding to the unsteady motions of an incompressible Maxwell fluid due to longitudinal and torsional oscillations of an infinite circular cylinder are presented as Fourier-Bessel series. The starting solutions that have been obtained, depending on initial and boundary conditions, are written as sum of the steady-state and transient solutions. They describe the motion of the fluid for same time after its initiation. After this time, when the transients disappear, these solutions tend to the steady-state solutions, which are periodic in time and independent of the initial conditions.

These solutions have been also written in simpler forms in terms of the modified Bessel functions $\mathrm{I}_0(\cdot),\ \mathrm{I}_1(\cdot),\ \mathrm{K}_0(\cdot)$ and $\mathrm{K}_1(\cdot)$. The numerical values as well as the diagrams corresponding to the steady-state solutions (26) and $(28)_1$, respectively, (27) and $(28)_2$, as it results from Figs. 1, are identical. The roots r_{1n} and r_{0n} have been approximated by $(4n+1)\pi/(4R)$ and $(4n-1)\pi/(4R)$, respectively.

Straightforward computations show that $w(r,t)$ and $\mathrm{v}(r,t)$, given by (18), (19), (24) and (25), satisfy both the associate partial differential equations (3) and (4) and all imposed initial and boundary conditions, the differentiation term by term in sums being clearly permissible. For $\lambda \to 0$ all solutions tend to those for Newtonian fluids. Eqs. (26), (27) and (28), for instance, become

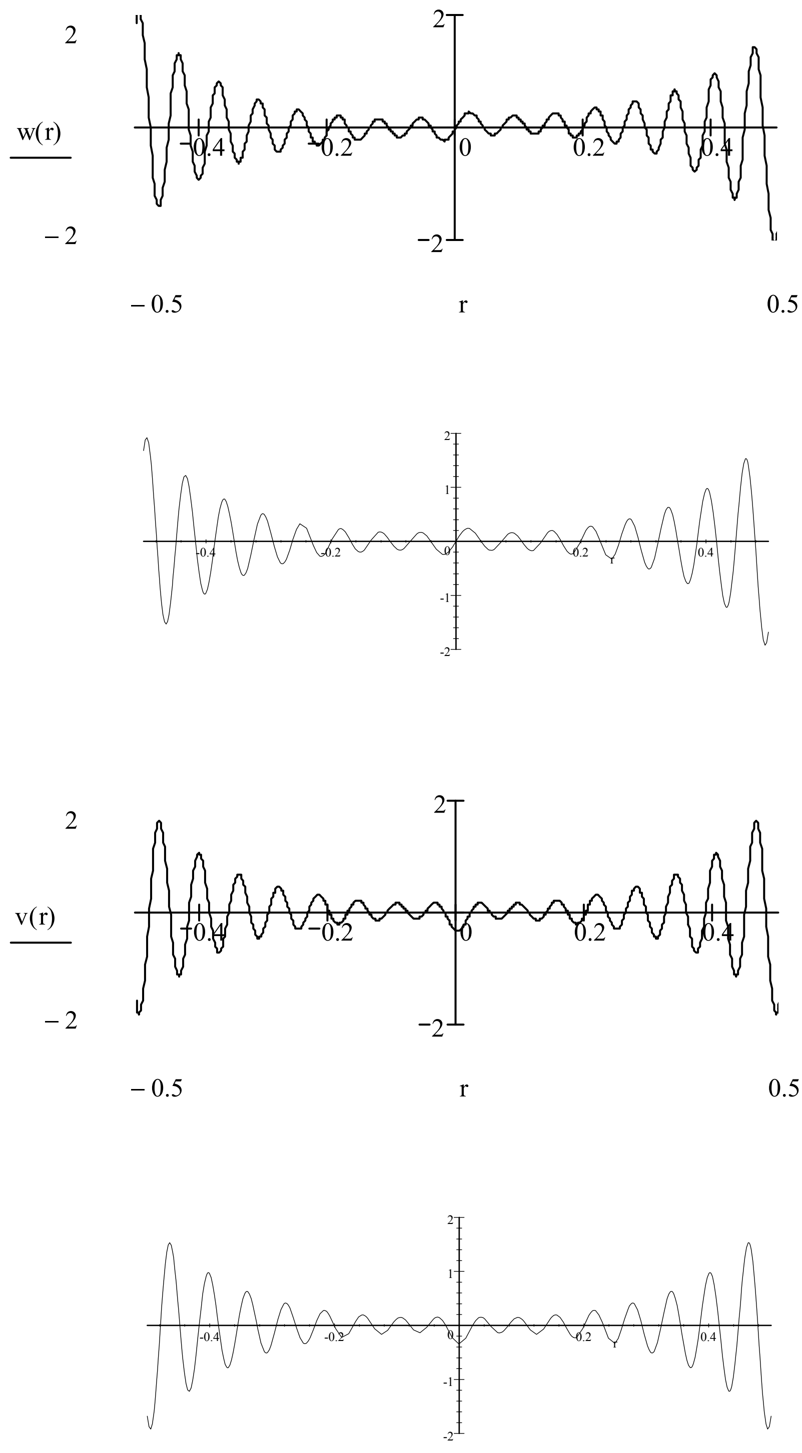
$$\begin{aligned} w_{st}(r,t) = {} & \frac{r}{R}W\cos(\omega_1 t) - \frac{2}{R}\left(\frac{\omega_1}{\nu}\right)^2 W\cos(\omega_1 t)\sum_{n=1}^{\infty}\frac{1}{r_{1n}^4 + (\omega_1/\nu)^2}\,\frac{\mathrm{J}_1(rr_{1n})}{r_{1n}\mathrm{J}_2(Rr_{1n})} + \\ & + \frac{2}{R}\frac{\omega_1}{\nu}W\sin(\omega_1 t)\sum_{n=1}^{\infty}\frac{r_{1n}}{r_{1n}^4 + (\omega_1/\nu)^2}\,\frac{\mathrm{J}_1(rr_{1n})}{\mathrm{J}_2(Rr_{1n})}, \end{aligned} \tag{29}$$

$$\begin{aligned} \mathrm{v}_{st}(r,t) = {} & V\cos(\omega_0 t) - \frac{2}{R}\left(\frac{\omega_0}{\nu}\right)^2 V\cos(\omega_0 t)\sum_{n=1}^{\infty}\frac{1}{r_{0n}^4 + (\omega_0/\nu)^2}\,\frac{\mathrm{J}_0(rr_{0n})}{r_{0n}\mathrm{J}_1(Rr_{0n})} + \\ & + \frac{2}{R}\frac{\omega_0}{\nu}V\sin(\omega_0 t)\sum_{n=1}^{\infty}\frac{r_{0n}}{r_{0n}^4 + (\omega_0/\nu)^2}\,\frac{\mathrm{J}_0(rr_{0n})}{\mathrm{J}_1(Rr_{0n})}, \end{aligned} \tag{30}$$

respectively,

$$w_{st}(r,t) = W\,\mathrm{Re}\left\{\frac{\mathrm{I}_1[(1+i)r\sqrt{\omega_1/(2\nu)}]}{\mathrm{I}_1[(1+i)R\sqrt{\omega_1/(2\nu)}]}\ \mathrm{e}^{\mathrm{i}\omega_1 t}\right\},\quad \mathrm{v}_{st}(r,t) = V\,\mathrm{Re}\left\{\frac{\mathrm{I}_0[(1+i)r\sqrt{\omega_0/(2\nu)}]}{\mathrm{I}_0[(1+i)R\sqrt{\omega_0/(2\nu)}]}\ \mathrm{e}^{\mathrm{i}\omega_0 t}\right\}, \tag{31}$$

In view of the asymptotic expansions for modified Bessel functions [6], $w_{st}(r,t)$ and $\mathrm{v}_{\mathrm{st}}(r,t)$ from (31) can be written in terms of the elementary functions sine, cosine, hyperbolic sine and hyperbolic cosine. However, the new approximations are valid only for $r \gg \sqrt{\nu/\omega_1}$, respectively, $r \gg \sqrt{\nu/\omega_0}$.



a. Eq. (26) with $W = 2$ and $\omega_1 = 2$

b. Eq. $(28)_1$ with $W = 2$ and $\omega_1 = 2$

c. Eq. (27) with $V = 2$ and $\omega_0 = 2$

d. Eq. $(28)_2$ with $V = 2$ and $\omega_0 = 2$

Fig. 1. Profiles of the steady-state velocities, corresponding to the motions within a circular cylinder, for $\nu = 0.0011746$ (glycerin), $\lambda = 3$, $R = 0.5$ and $t = 5$s.

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Received December 7, 2005