THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A
Volume 8 , Number 1/2007, pp.000-000

MULTIOBJECTIVE NONLINEAR FRACTIONAL PROGRAMMING PROBLEMS INVOLVING GENERALIZED d- TYPE-I n-SET FUNCTIONS

Ioan M. STANCU-MINASIAN

The Romanian Academy, Institute of Mathematical Statistics and Applied Mathematics,
Calea 13 Septembrie nr. 13, 050711, Bucharest 5, Romania,
E-mail: stancu_minasian@yahoo.com

We establish duality results under generalized convexity assumption for a multiobjective nonlinear fractional programming problem involving generalized *d*-type-I *n*-set functions .

Key words: duality, multiobjective programming, fractional programming, *n*-set functions, generalized d-type-I functions.

1. PRELIMINARIE

In this section we introduce the notation and definitions which will be used throughout the paper.

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space and $\mathbb{R}^m_+$ its positive orthant, i.e.

$$\mathbb{R}^m_+ = \left\{x = (x_j) \in \mathbb{R}^m, x_j \geqq 0, j = 1,\dots,m\right\}.$$

For $x = (x_1,\dots,x_m),\ y = (y_1,\dots,y_m) \in \mathbb{R}^m$ we put $x \leqq y$ iff $x_i \leqq y_i$ for each $i \in M = \{1,2,\dots,m\};$ $x \le y$ iff $x_i \leqq y_i$ for each $i \in M$, with $x \ne y;$ $x < y$ iff $x_i < y_i$ for each $i \in M$. We write $x \in \mathbb{R}^m_+$ iff $x \geqq 0$.

For an arbitrary vector $x \in \mathbb{R}^n$ and a subset J of the index set $\{1,2,\dots,n\}$, we denote by x_J the vector with components $x_j,\ j \in J$.

Let (X,Γ,μ) be a finite non-atomic measure space, and let d be the pseudometric on Γ^n defined by

$$d(S,T) = \left[\sum_{k=1}^{n} \mu^2 (S_k \Delta T_k)\right]^{1/2}$$

for $S = (S_1,\dots,S_n), T = (T_1,\dots,T_n) \in \Gamma^n$, where Γ^n is the *n*-fold product of a σ-algebra Γ of subsets of a given set *X*, and Δ denotes the symmetric difference. Thus (Γ^n, d) is a pseudometric space, which will serve as the domain for most of the functions that will be used in this paper.

For $h \in L_1(X,\Gamma,\mu)$, the integral $\int_S h\, \mathrm{d}\mu$ will be denoted by $\langle h, I_S \rangle$, where I_S is the indicator (characteristic) function of $S \in \Gamma$.

We next introduce the notion of differentiability for *n*-set functions. This was originally introduced by Morris [4] for set functions and subsequently extended by Corley [1] to n-set functions.

A function $\varphi : \Gamma \to \mathbb{R}$ is said to be differentiable at $S^0 \in \Gamma$ if there exists $D\varphi(S^0) \in L_1(X,\Gamma,\mu)$, called the derivative of φ at S^0, and $\psi : \Gamma \times \Gamma \to \mathbb{R}$ such that for each $S \in \Gamma$,

$$\varphi(S) = \varphi(S^0) + \langle D\varphi(S^0), I_S - I_{S^0} \rangle + \psi(S, S^0)$$

where $\psi(S,T)$ is $o\left(d\left(S,S^0\right)\right)$, that is $\lim\limits_{d\left(S,S^0\right)\to 0}\psi\left(S,S^0\right)/d\left(S,S^0\right)=0$, and d is a pseudometric on Γ [4].

A function $F:\Gamma^n\to\mathbb{R}$ is said to have a partial derivative at $S^0=\left(S_1^0,\ldots,S_n^0\right)$ with respect to its k-th argument, $1\leqq k\leqq n$, if the function

$$\varphi(S_k)=F\left(S_1^0,\ldots,S_{k-1}^0,S_k,S_{k+1}^0,\ldots,S_n^0\right)$$

has derivative $D\varphi(S_k^0)$, and we define $D_kF\left(S^0\right)=D\varphi\left(S_k^0\right)$. If the $D_kF\left(S^0\right)$, $1\leqq k\leqq n$, all exist, then we put $DF\left(S^0\right)=\left(D_1F\left(S^0\right),\ldots,D_nF\left(S^0\right)\right)$. If $H:\Gamma^n\to\mathbb{R}^m$, $H=(H_1,\ldots,H_m)$, we put $D_kH\left(S^0\right)=\left(D_kH_1\left(S^0\right)\right)$.

A function $F:\Gamma^n\to\mathbb{R}$ is said to be differentiable at S^0 if there exist $DF\left(S^0\right)$ and $\psi:\Gamma^n\times\Gamma^n\to\mathbb{R}$ such that

$$F(S)=F\left(S^0\right)+\sum_{k=1}^{n}\left\langle D_kF\left(S^0\right),I_{S_k}-I_{S_k^0}\right\rangle+\psi\left(S,S^0\right),$$

where $\psi\left(S,S^0\right)$ is $o\left[d\left(S,S^0\right)\right]$ for all $S\in\Gamma^n$.

A vector set function $f=\left(f_1,\ldots,f_p\right):\Gamma\to\mathbb{R}^p$ is differentiable on Γ if all its component functions f_i, $1\leqq i\leqq p$, are differentiable on Γ.

Consider the multiobjective nonlinear fractional programming problem involving n-set functions.

$$\text{(P)}\qquad \text{minimize}\left\{F(S)=\left(\frac{F_1(S)}{G_1(S)},\ldots,\frac{F_p(S)}{G_p(S)}\right)\right\},$$
$$\text{subject to } H_j(S)\leqq 0,\ j\in M,\ S=(S_1,\ldots,S_n)\in\Gamma^n$$

where $F_i, G_i, i\in P=\{1,2,\ldots,p\}$, and H_j, $j\in M$ are differentiable real valued functions defined on Γ^n with

$$F_i(S)\geq 0 \text{ and } G_i(S)>0, \text{ for all } i\in P.$$

The term "minimize" being in Problem (P) is for finding efficient, and weak efficient solutions. Let $S_0=\left\{S \middle| S\in\Gamma^n, H(S)\leq 0\right\}$ be the set of all feasible solutions to (P), where $H=(H_1,\ldots,H_m)$.

A feasible solution S^0 to (P) is said to be an efficient solution to problem (P) if there exists no other feasible solution S to (P) such that $F_i(S)\leqq F_i(S^0)$, for all $i\in P$, with strict inequality for at least $i\in P$.

A feasible solution S^0 to (P) is said to be a weakly efficient solution to problem (P) if there exists no other feasible solution S to (P) such that $F_i(S)<F_i(S^0)$, for all $i\in P$.

Let $\rho_1,\ldots,\rho_p,\rho_1',\ldots,\rho_m'$, ρ,ρ' be real numbers and put $\overline{\rho}=(\rho_1,\ldots,\rho_p)$ and $\overline{\rho}'=(\rho_1',\ldots,\rho_m')$. Also let $\theta:\Gamma^n\times\Gamma^n\to\mathbb{R}_+$ be a function such that $\theta(S,S^0)\neq 0$ for $S\neq S^0$.

Along the lines of Jeyakumar and Mond [2] and Suneja and Srivastava [7], Preda, Stancu-Minasian and Koller [5] defined new classes of n-set functions, called $(\overline{\rho},\overline{\rho}',d)$-type-I, (ρ,ρ',d)-quasi type-I, (ρ,ρ',d)-pseudo type-I, (ρ,ρ',d)-quasi-pseudo type-I, (ρ,ρ',d)-pseudo-quasi type-I.

Definition 1.1. [5] *We say that* (F,H) *is of* $(\bar{\rho},\bar{\rho}',d)$ *- type-* I *at* $S^0 \in \Gamma^n$ *if there exist functions* $\alpha_i,\beta_j : \Gamma^n \times \Gamma^n \to \mathbb{R}_+ \setminus \{0\}, i \in P, j \in M$, *such that for all* $S \in S_0$, *we have*

$$F_i(S) - F_i(S^0) \geqq \alpha_i(S,S^0) \sum_{k=1}^{n} \left\langle D_k F_i(S^0), I_{S_k} - I_{S_k^0} \right\rangle + \rho_i \theta(S,S^0),\ i \in P \tag{2}$$

and

$$-H_j(S_0) \geqq \beta_j(S,S^0) \sum_{k=1}^{n} \left\langle D_k H_j(S^0), I_{S_k} - I_{S_k^0} \right\rangle + \rho'_j \theta(S,S^0),\ j \in M. \tag{3}$$

We say that (S,H) is of $(\bar{\rho},\bar{\rho}',d)$-semistrictly type-I at S^0 if in the above definition we have $S \neq S^0$ and (2) is a strict inequality.
Now, we introduce

Definition 1.2. [8] A feasible solution S^0 to (P) is said to be a regular feasible solution if there exists $\hat{S} \in \Gamma^n$ such that

$$H_j(S^0) + \sum_{k=1}^{n} \left\langle D_k H_j(S^0), I_{\hat{S}_k} - I_{S_k^0} \right\rangle < 0,\ j \in M.$$

Now, for each $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{R}^p_+$ we consider the parametric problem

(P_λ) minimize $\left(F_1(S) - \lambda_1 G_1(S), \dots, F_p(S) - \lambda_p G_p(S)\right)$.

subject to

$$H_j(S) \leqq 0,\ j \in M, S = (S_1, \dots, S_n) \in \Gamma^n$$

It is well known that (P_λ) is closely related to problem (P).

The following lemma is well known in fractional programming.

Lemma 1.3. *An* S^0 *is an efficient solution to* (P) *if and only if is an efficient solution to* (P_{λ^0}) *with* $\lambda_i^0 = F_i(S^0)/G_i(S^0), i = 1,2,\dots,p.$

In this paper, the proofs of the duality results for Problem (P) will invoke the following necessary optimality conditions (see Zalmai [8], Theorems 3.1 and 3.2 and Corley [1], Theorem 3.7.)]

Theorem 1.4. *Let* S^0 *be a regular efficient* (*or weakly efficient*) *solution to* (P) *and assume that* $F_i, G_i, i \in P$, *and* $H_j, j \in M$, *are differentiable at* S^0. *Then there exist* $u^0 \in \mathbb{R}^p_+, \sum_{i=1}^{p} u_i^0 = 1, v^0 \in \mathbb{R}^p_+$ *and* $\lambda^0 \in \mathbb{R}^p_+$ *such that*

$$\sum_{k=1}^{n} \left\langle \sum_{i=1}^{p} u_i^0 (D_k F_i(S^0) - \lambda_i^0 D_k G_i(S^0)) + \sum_{j=1}^{m} v_j^0 D_k H_j(S^0), I_{S_k} - I_{S_k^0} \right\rangle \geqq 0 \ \textit{for all}\ S \in \Gamma^n, \tag{4}$$

$$u_i^0 \left(F_i(S^0) - \lambda_i^0 G(S^0)\right) \geqq 0,\ i \in \mathrm{P}, \tag{5}$$

$$v_j^0 H_j(S^0) = 0,\ j \in M. \tag{6}$$

2. DUALITY

In this section, in the differentiable case, based on the equivalence of (P) and P_λ a dual for P_λ is defined and some duality results in $(\overline{\rho}, \overline{\rho}', d)$-type-I assumptions are stated. With P_λ we associate a dual stated as

$$(\mathrm{D}) \qquad \text{maximize } (\lambda_1, \ldots, \lambda_p)$$

subject to

$$\sum_{i=1}^{p} \sum_{k=1}^{n} u_i \left\langle D_k F_i(T) - \lambda_i D_k G_i(T), I_{S_k} - I_{S_k^0} \right\rangle + \sum_{j=1}^{m} \sum_{k=1}^{n} v_j \left\langle D_k H_j(T), I_{S_k} - I_{S_k^0} \right\rangle \geqq 0,\ S \in \Gamma^n, \tag{7}$$

$$u_i \left(F_i(T) - \lambda_i G(T) \right) \geqq 0, i \in P, \tag{8}$$

$$v_j H_j(T) \geqq 0,\ j \in M. \tag{9}$$

$$u \in \mathbb{R}_+^p, \sum_{i=1}^{p} u_i = 1, v \in \mathbb{R}_+^m, \lambda \in \mathbb{R}_+^p. \tag{10}$$

Let D_θ be the set of feasible solutions to (D).

Theorem 2.1. (Weak duality). *Let* (T, u, v, λ) *be a feasible solution to problem* (D) *and assume that*

(i_1) *for each* $i \in P$ *and* $j \in M$, $\left(F_i(\cdot) - \lambda_i G_i(\cdot), H_j(\cdot) \right)$ *is of* $(\overline{\rho}, \overline{\rho}', d)$*-type-*I *at T.*

We also assume that any of the following conditions hold:

(i_2) $u_i > 0$ *for any* $i \in P$, $\sum_{i=1}^{p} \frac{u_i \rho_i}{\alpha_i(S,T)} + \sum_{j=1}^{m} \frac{v_j \rho_j'}{\beta_j(\mathrm{S},\mathrm{T})} \geqq 0$ *and for some* $i \in P$ *and* $j \in M$;

$\left(\mathrm{F}_i(\cdot) - \lambda_i \mathrm{G}_i(\cdot), \mathrm{H}_j(\cdot) \right)$ *is of* $(\overline{\rho}, \overline{\rho}', d)$*-semistrictly type-*I *at* T;

(i_3) $\sum_{i=1}^{p} \frac{u_i \rho_i}{\alpha_i(\mathrm{S},\mathrm{T})} + \sum_{j=1}^{m} \frac{v_j \rho_j'}{\beta_j(\mathrm{S},\mathrm{T})} > 0.$

Then for any $\mathrm{S} \in S_\theta$ *one cannot have*

$F_i(S)/G_i(S) \leqq \lambda_i$ *for any* $i \in P$,

$F_j(S)/G_j(S) \leqq \lambda_j$ *for some* $j \in P$

Corollary 2.2. *Let* S^0 *and* $(S^0, u^0, v^0, \lambda^0)$ *be feasible solutions to* (P_{λ^0}) *and* (D), *respectively. If the hypotheses of Theorem* 2.1 *are satisfied, then* S^0 *is an efficient solution to* (P_{λ^0}) *and* $(S^0, u^0, v^0, \lambda^0)$ *is an efficient solution to* (D).

Theorem 2.3. (Strong duality). *Let* S^0 *be a regular efficient solution to* (P). Then there exist $u^0 \in \mathbb{R}_+^p, \sum_{i=1}^{p} u_i^0 = 1, v^0 \in \mathbb{R}_+^m$ *and* $\lambda^0 \in \mathbb{R}_+^p$, *such that* $(S^0, u^0, v^0, \lambda^0)$ *is a feasible solution to* (D).

Further, if the conditions of the weak duality Theorem 2.1 *also hold, then* $\left(S^0, u^0, v^0, \lambda^0\right)$ *is an efficient solution to* (D).

Now we give a strict converse duality theorem of Mangasarian type [3] for (P_{λ^0}) and (D).

Theorem 2.4. (Strict converse duality). *Let* S^* *and* $\left(S^0, u^0, v^0, \lambda^0\right)$ *be efficient solutions to* (P_{λ^0}) *and* (D), *respectively. Assume that*

(j_1) $$\sum_{i=1}^{p} u_i^0 \left(F_i(S^*) - \lambda_i^0 G_i(S^*)\right) \leqq \sum_{i=1}^{p} u_i^0 \left(F_i(S^0) - \lambda_i^0 G_i(S^0)\right);$$

(j_2) *for any* $i \in P$ *and* $j \in M$, $\left(F_i(\cdot) - \lambda_i G_i(\cdot), H_j(\cdot)\right)$ *is of* $\left(\overline{\rho}, \overline{\rho}', d\right)$- *semistrictly type* $-\mathrm{I}$ *at* T;

(j_3) $$\sum_{i=1}^{p} \frac{u_i \rho_i}{\alpha_i (S,T)} + \sum_{j=1}^{m} \frac{v_j \rho_j'}{\beta_j (S,T)} > 0.$$

Then $\mathrm{S}^0 = \mathrm{S}^*$.

The proofs will appear in [6].

REFERENCES

1. CORLEY, H.W., *Optimization theory for n-set functions*, J. Math. Anal. Appl., **127** (1987), 193–205.
2. JEYAKUMAR, V., MOND, B., *On generalized convex mathematical programming*, J. Austral. Math. Soc. Ser. B, **34** (1992), 43–53.
3. MANGASARIAN, O.L., *Nonlinear Programming*, McGraw-Hill, New York, 1969.
4. MORRIS, R.J.T., *Optimal constrained selection of a measurable subset*, J. Math. Anal. Appl., **70** (1979) 2, 546–562.
5. PREDA, V., STANCU-MINASIAN, I.M., KOLLER, E., *On optimality and duality for multiobjective programming problems involving generalized d-type-*I *and related n-set functions*. J.Math.Anal.Appl. **283**(2003)1, 114-128.
6. STANCU-MINASIAN, I.M., *Duality for multiobjective fractional programming problems involving generalized d-type-* I *n-set* (Submitted).
7. SUNEJA, S.K., SRIVASTAVA, M.K., *Optimality and duality in nondifferentiable multiobjective optimization involving d-type* I *and related functions,* J.Math.Anal.Appl., **206**(1997)2, 465-479.
8. ZALMAI, G.J., *Optimality conditions and duality for multiobjective measurable subset selection problems*, Optimization, **22** (1991) 2, 221–238.

Received February 1, 2007