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SPINS STATES WITH MAXIMAL CLASSICAL ENTROPY [1]

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In the present paper we shall give the numerical values of the classical entropy of some quantum states of a given spin 2j. These calculations support the idea that the states with the most spread Majorana configurations have the maximal values of classical entropy.

1 INTRODUCTION

In the Majorana's sphere representation [1, 10], a spin state of definite spin j is represented by $2j$ points on the Riemann sphere, generalizing the Bloch representation for spin $j=\frac{1}{2}$. Majorana introduced this sphere representation to aid the calculations of transition probabilities in quantum mechanics. In the paper [2], Lieb have made the conjecture that the states with minimum value of the classical entropy are the coherent states associated with the representations of the rotations group [3, 4]. In the paper [5] we have proved this conjecture for the states with spin $j=1$. Because the fact that the classical entropy of a spin state is a functional which is invariant with respect to the rotations, and as a consequence of the L. Michel conjecture [6, 7], we try to "prove" that the states which give the maximum value for the classical entropy are the states which correspond to the critical orbits.

2. THE MAJORANA SPHERE

In order to give a classification of the orbits of the unitary action U_j of the rotation group $SO(3)$ in the $2j+1-$dimensional Hilbert space H_j we shall use the Majorana representation of the spin states. In this representation there is a one-to-one, covariant correspondence between the vectors from the Hilbert space H_j and the sets of $2j-$points on the two-dimensional Majorana's sphere S^2. Let $u \in H_j$ denotes an abstract quantum state with definite integer or half integer spin j i.e. it is an eigenvector of the total angular momentum $\vec{J}^2$. With respect to a certain coordinate frame u may be written $|u>=\sum_{m=j}^{m=-j} c_m v_m$ where v_m is an eigenvector of the angular momentum component operator $\vec{J}_z$ with eigenvalue m. The phase convention used is the standard Condon-Shortley convention given by applying successive raising operators to the vector v_{-j}. In order to describe this correspondence let us consider the two-dimensional sphere S^2 as the

[1] This paper is the internal report in the **NIPNE** contract Nr. **63-78-45 (1**978). The conjecture of L. Michel was the principal motivation of these calculations. The above result where presented at a seminar held at the Theoretical
Physics Department of the Physics Faculty of the University from Bucharest Professor Louis Michel was one of the participants. I dedicate this paper to the memory of Louis Michel.

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Riemann's sphere with the complex coordinate $z(\theta,\phi)=\exp(i\phi)\tan\frac{\theta}{2}$ obtained by the stereographic projection. The action of the rotation group $SO(3)$ on the coordinate z is given by the homographic transformations. Let K_j be the Hilbert space of the polynomials in $z-$variable of degree smaller or equal with $2j$ and square-integrable with respect to the invariant measure on sphere $d\nu(z,\bar{z})=\frac{2j+1}{2\pi i}\frac{dz\Lambda d\bar{z}}{(1+|z|^2)^{2j+1}}$. On the Hilbert space K_j with dimension equal with $2j+1$ and the canonical orthonormal basis given by the polynomials $\{\left(\sqrt{\frac{(2j)!}{(j+m)!(j-m)!}}z^{j+m}\right), m=-j,-j+1,...,j-1,j\}$ the rotation group $SO(3)$ acts by unitary operators defined in the following way:

$$(V_j(g)P)(z)=P(\frac{az+b}{-\bar{b}z+\bar{a}})(-\bar{b}z+\bar{a})^{2j} \tag{2.1}$$

Then the following theorem holds:

THEOREM: *The intertwining operator $T:H_j\to K_j$ for the equivalent irreducible unitary representations U_j and V_j of $SO(3)$ is given by:*

$$(Tu)(z)=P_u(z)=(u,U_j(g)v_j)(1+|z|^2)^j \tag{2.2}$$

where $z=\frac{b}{\bar{a}}$ and $u\in H_j$.

Hence, this theorem ascribes to any vector, $u\in H_j$, a polynomial $P_u(z)\in K_j$. If $|u>=\sum_{m=j}^{m=-j}c_m v_m$ then:

$$P_u(z)=\sum_{m=-j}^{m=j}(-1)^{j+m}\bar{c}_m\sqrt{\frac{(2j)!}{(j+m)!(j-m)!}}z^{j+m} \tag{2.3}$$

Since $\sum_{m=-j}^{m=j}|c_m|^2=1$ every polynomial $P_u(z)$ is uniquely determined up to a phase by its $2j$ roots $\{z_k\}_{k=1,2,...,2j}$ of which are $2j$ points on the Riemann's sphere. This representation is called the Majorana representation. The Majorana representation is covariant with respect of the actions of the rotation group $SO(3)$ on the Hilbert space K_j and the sphere S^2 respectively. This means that to the vector $U_j(g)u$ the Majorana representation ascribes the set $\{z_k\}_{k=1,2,...,2j}$ of $2j$ points. This property is a consequence of the fact that the operator T is an intertwining operator i.e. $TU_j(g)=V_j(g)T$ and of the fact that:

$$(V_j(g)P)(z)=(-\bar{b}z+\bar{a})P(gz) \tag{2.4}$$

As a consequence the problem of the classification of the orbits of the rotation group $SO(3)$ in H_j is reduced to the problem of the classification of the orbits of the action of the rotation group on the configurations of $2j$ points on the Majorana's sphere S^2.

3. THE SUBGROUPS OF THE ROTATIONS GROUP WHICH LEAVE UNCHANGED SOME MAJORANA'S CONFIGURATIONS OF $2j$ POINTS

The subgroups of the rotations group which leave unchanged some configurations of $2j$ points on the Majorana's sphere are the following subgroups: $SO(2), O(2), C(n), D(n), (n \le 2j), T, O, I$.

The constellations which describe the critical orbits [6, 8] are given in the following table:

Table 1

The constellations	The associated vectors	The symmetry group
$(n_N, 0, n_S)$ $m = \frac{n_N - n_S}{2}$	v_m v_{-m}	$SO(2)$
$(j, 0, j)$	v_0	$O(2)$
$(j-m, 2m, j-m)$	$\frac{1}{\sqrt{2}}(v_m + v_{-m})$	$D(2m)$
$z^k(z^3 - 2\sqrt{2})^k = 0$, $j = 2k, k = 1,2,...$	$\frac{1}{\sqrt{3}}v_2 + \sqrt{\frac{2}{3}}v_{-1}$ *only for* $j = 2$	T tetrahedral
$z^k(z^4 - 1)^k \frac{1}{z^k} = 0$, $j = 3k$ *or* $4k, k = 1,2,...$	$\frac{1}{\sqrt{2}}(v_2 + v_{-2})$ *only for* $j = 3$	O octahedral
$z^k(z^5 - (\frac{\sqrt{5}-1}{2})^5)^k (z^5 + (\frac{2}{\sqrt{5}-1})^5) \frac{1}{z^k} = 0$, $j = 6k$ *or* $10k, k = 1,2,...$		I icosahedral

We have used the following notations for the configurations of $2j$ points which describe the critical orbits: (n_N, n_E, n_S), where n_N is the number of points which are placed at the north pole, n_E is the number of points which are placed on the equator and n_S is the number of points which are placed at the south pole. Evidently we must have $n_N + n_E + n_S = 2j$.

4. THE CLASSICAL ENTROPY OF THE STATES DESCRIBED BY THE VECTORS OF THE TYPE $c_m v_m + c_n v_n$

In this section, we shall give the analytic formulae for the classical entropy $S^j(u)$ of the spin system state $u = c_m v_m + c_n v_n$ with real components c_m, c_n and with $m > n$, as a function of the positive real parameter r defined by:

$$r = \left[\frac{c_m{}^2 (j+n)!(j-n)!}{c_n{}^2 (j+m)!(j-m)!} \right]^{\frac{1}{m-n}} \in (0, \infty) \tag{4.1}$$

In order to simplify the exposition we shall not give the calculations which are somewhat laborious. Because the vector $u = c_m v_m + c_n v_n$ is completely determined by m, n, r we shall denote by $S^j(m, n; r)$ the classical entropy of this quantum state.

In the following we shall give the concrete values of this general formula for the classical entropy for the states enumerate in the Table1:

$$\begin{aligned} S^j(v_m) &= (j+m)(\frac{1}{j+m+!} + \ldots + \frac{1}{2j}) + (j+m)(\frac{1}{j-m+1} + \ldots + \frac{1}{2j}) - \\ &\ln \frac{(2j)!}{(j+m)!(j-m)!} + \frac{2j}{2j+1} \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &S^j(2,-1;r) = -2\ln 2 - 3\ln(1+r) + \ln(1+4r^3) - 2 + \\ &\frac{1}{30(1+4r^3)(1+r)^5}[336r^8 + 2040r^7 + 4980r^6 + 6334r^5 + 4850r^4 + 2941r^3 + \\ &1795r^2 + 785r + 139] \end{aligned} \tag{4.3}$$

From the formula for $S^j(m,-m;r)$ we obtain that

$$\frac{d}{dr} S^j(m,-m;r)|_{r=1} = 0 \tag{4.4}$$

and

$$\frac{d^2}{dr^2} S^j(m,-m;r)|_{r=1} < 0 \tag{4.5}$$

This proves that $S^j(m,-m;r)$ takes the maximum value on the vectors $\frac{1}{\sqrt{2}}(v_m + v_{-m})$ which gives the critical orbits with the symmetries $D(2m)$ in general and with the symmetry O for $j = 3$ and $m = 2$. Also it is easy to shown that:

$$\frac{d}{dr} S^2(2,-1;r)|_{r=\frac{1}{2}} = 0 \tag{4.6}$$

and

$$\frac{d^2}{dr^2} S^j(2,-1;r)|_{r=\frac{1}{2}} < 0 \tag{4.7}$$

i.e. among the vectors $u = c_2 v_2 + c_{-1} v_{-1}$ the maximum value of the classical entropy is attained on the vector $\frac{1}{\sqrt{3}}(v_2 + \sqrt{2} v_{-1})$ with the symmetry T. For the "crown states" [9] $u = c_j v_j + c_{-j} v_{-j}$ we have:

$$S^j(j,-j;r)=\frac{2j}{2j+1}+\ln(1+r^{2j})2j\ln(1+r)+$$
$$\frac{2j}{1+r^{2j}}\sum_{k=1}^{2j+1}\frac{r^k+r^{2j}}{k(1+r)^k}-\left(\frac{2j}{2j+1}+2\right)\frac{r^{2j}}{(1+r)^{2j}(1+r^{2j})} \tag{4.8}$$

In the particular case of the vectors $u=\frac{1}{\sqrt{2}}(v_j+v_{-j})$ becomes:

$$S^j(j,-j;1)=\frac{2j}{2j+1}+\ln 2+2j\left(\sum_{k=1}^{2j-1}\frac{1}{2^k k}-\ln 2\right) \tag{4.9}$$

We remark that

$$S^j(j,-j;1)>\frac{2j}{2j+1} \tag{4.10}$$

is a consequence of the fact that always:

$$\ln 2+2j\left(\sum_{k=1}^{2j-1}\frac{1}{2^k k}-\ln 2\right)>0 \tag{4.11}$$

In the case $j=1$ we obtain for the states of type $\frac{1}{\sqrt{r+2}}(\sqrt{r}\,v_0+\sqrt{2}\,v_{-1})$

$$S^1(0,-1;r)>\frac{2}{3}+\frac{r}{r+2}-\ln(1+\frac{r}{r+2}) \tag{4.12}$$

Hence $S^1(0,-1;r)$ is a monotone increasing function of $r\in(0,\infty)$ with minimum value $S^1(0,-1;0)=\frac{2}{3}$ attained for the vector v_1 and any rotationally related other vector. Because in the case $j=1$ we have only three strata generated by vectors v_1, v_1 and $\frac{1}{\sqrt{r+2}}(\sqrt{r}\,v_0+\sqrt{2}\,v_{-1})$ it follows that the Lieb's conjecture is valid [5]. In the table 2 we shall give numerical values of the classical entropy. We remark that in the case $j=2$ the vectors $\frac{1}{\sqrt{2}}(v_1+v_{-1})$ and $\frac{1}{\sqrt{2}}(v_2+v_{-2})$ are in the same orbits.

Table 2

j	u	$S^j(u)-\frac{2j}{2j+1}$	The symmetry group of u
1	2	3	4
$\frac{3}{2}$	$v_{\frac{3}{2}}$	0	SO(2)
	$v_{\frac{1}{2}}$	0.40139	SO(2)
	$\frac{1}{\sqrt{2}}(v_{\frac{1}{2}}+v_{-\frac{1}{2}})$	0.27632	D(1)
	$\frac{1}{\sqrt{2}}(v_{\frac{3}{2}}+v_{-\frac{3}{2}})$	0.48870	D(2)
2	v_2	0	SO(2)
	v_1	0.44703	SO(2)
	v_0	0.54157	O(2)
	$\frac{1}{\sqrt{2}}(v_1+v_{-1})$	0.58721	D(4)
	$\frac{1}{\sqrt{2}}(v_2+v_{-2})$	0.58721	D(4)
	$\frac{1}{\sqrt{3}}(v_2+\sqrt{2}v_{-1})$	0.69166	T
$\frac{5}{2}$	$v_{\frac{3}{2}}$	0	SO(2)
	$v_{\frac{3}{2}}$	0.47389	SO(2)
	$v_{\frac{1}{2}}$	0.61408	SO(2)
	$\frac{1}{\sqrt{2}}(v_{\frac{1}{2}}+v_{-\frac{1}{2}})$	0.68698	D(1)
	$\frac{1}{\sqrt{2}}(v_{\frac{3}{2}}+v_{-\frac{3}{2}})$	0.82195	D(3)
	$\frac{1}{\sqrt{2}}(v_{\frac{5}{2}}+v_{-\frac{5}{2}})$	0.63880	D(5)

1	2	3	4
3	v_3	0	SO(2)
	v_2	0.49157	SO(2)
	v_1	0.65861	SO(2)
	v_0	0.70426	O(2)
	$\frac{1}{\sqrt{2}}(v_1 + v_{-1})$	0.70505	D(2)
	$\frac{1}{\sqrt{2}}(v_2 + v_{-2})$	0.97879	O
	$\frac{1}{\sqrt{2}}(v_3 + v_{-3})$	0.66550	D(6)
$\frac{7}{2}$	$v_{\frac{3}{2}}$	0	SO(2)
	$v_{\frac{3}{2}}$	0.50408	SO(2)
	$v_{\frac{3}{2}}$	0.68820	SO(2)
	$v_{\frac{1}{2}}$	0.76131	SO(2)
	$\frac{1}{\sqrt{2}}(v_{\frac{1}{2}} + v_{-\frac{1}{2}})$	0.89933	D(1)
	$\frac{1}{\sqrt{2}}(v_{\frac{3}{2}} + v_{-\frac{3}{2}})$	0.93844	D(2)
	$\frac{1}{\sqrt{2}}(v_{\frac{5}{2}} + v_{-\frac{5}{2}})$	1.07783	D(3)
	$\frac{1}{\sqrt{2}}(v_{\frac{7}{2}} + v_{-\frac{7}{2}})$	0.67901	D(7)

REFERENCES

1. MAJORANA, E., Nuovo Cimento. **57,** 43,(1932).
2. LIEB, E. H., Commun. Math. Phys. **62**, 35, (1978).
3. KIRILLOV, A., *Elements de la Theorie des Representations*, Mir, Moscow, 1974.
4. SCUTARU, H., Rep. Math. Phys. **15**, 305 (1977).
5. SCUTARU, H., *On Lieb's entropy conjecture*, Preprint FT-180-1979, and arXive-math-ph/9909024, (Published in Rom. J. Phys. **47,** 189 (2002)).
6. MICHEL, L., J. Math. Phys. **15**, 5601, (1998).
7. MICHEL, L., and RADICATI, L., A., Ann. Phys., (N.Y), **66**, 758, (1971).
8. BACRY, H., J. Math. Phys. **15**, 1686, (1974).
9. BACRY, H., Phys. Rev. **18**, 617, (1978).
10. BACRY, H., Commun. Math. Phys. **72,** 119, (1980).

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