ON INFINITE BERNOULLI CONVOLUTIONS

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Let \((X_n)_{n \geq 0}\) be a sequence of i.i.d. nondegenerate integrable random variables and let \(q \in [0, 1]\). Let \(S(n,q) = X_0 + qX_1 + \ldots + q^nX_n\). If \(|q| < 1\) the sequence \((S(n,q))_{n \geq 0}\) is almost surely convergent to an integrable random variable \(S(q)\) which has a distribution denoted by \(\mu(q)\). Even in the most simple case when \(X_n \sim \text{Binomial}(1, \frac{1}{2})\) behaves mysteriously erratic when \(q \in (\frac{1}{2}, 1)\). We prove that there still exists a regularity, namely

\[
0 < q < \frac{1}{2} \implies \text{Uniform}(0, L)
\]

and

\[
\frac{1}{2} < q < 1 \implies \mu(q) \leq \text{Uniform}(0, L),
\]

where \(1/L = 1 - q\) and “\(\leq\)” is the Choquet convex domination. The problem has a clear financial motivation: if \(q\) is an actualization factor, then \(S(q)\) is the actual value of the infinite sum \(X_0 + X_1 + \ldots\).

1. THE PROBLEM

Let \((X_n)_{n \geq 0}\) be a sequence of i.i.d. random variables and \(S(n,q) = X_0 + qX_1 + \ldots + q^nX_n\), with \(q\) a real number. As

\[
S(n+k,q) - S(n,q) - q^{n+1}S(k-1,q)
\]

(1.1)

(meaning that \((S(n,q))_{n \geq 0}\) is Cauchy in \(L^1\), hence convergent in \(L^1\)). It is easy to check that it is also convergent a.s. since the series \(S(q) = \sum_{n=0}^{\infty} q^nX_n\) converges a.s. If \(X_n \in L^\infty\), the convergence is even uniform.

The real problem is to compute the distribution of \(S(q)\). Let \(F_q\) and \(F_{n,q}\) be the distribution functions of \(S(q)\) and \(S(n,q)\). Let also \(\nu\) be the distribution of \(X_n\) and \(\mu(q), \mu(n,q)\) the distributions of \(S(q)\) and \(S(n,q)\).

If \(|X_n| \leq M\) a.s. (that is, \(X_n\) are essentially bounded), it is easy to see that
\[ S(n, q) - \frac{|q|^{n+1}M}{1-|q|} \leq S(q) \leq S(n, q) + \frac{|q|^{n+1}M}{1-|q|} \]  

Therefore, a coarse evaluation of \( F_q \) would be

\[ F_{n,q}(x - \frac{|q|^{n+1}M}{1-|q|}) \leq F_q(x) \leq F_{n,q}(x + \frac{|q|^{n+1}M}{1-|q|}) \]

which, for great \( n \), is good enough for continuity points of \( F_{n,q} \).

Anyway, estimation (1.4) is useless if we want to know the type of the distribution \( \mu(q) \). According to the Lebesgue – Nikodym theorem any probability distribution \( \mu \) on the real line can be written as a mixture

\[ \mu = \alpha \mu_d + \beta \mu_{SC} + \gamma \mu_{AC} \]  

where \( \alpha, \beta, \gamma \geq 0, \alpha + \beta + \gamma + 1, \mu_d \) is a discrete distribution, \( \mu_{SC} \) is continuous but singular (i.e. there exists a Borel set \( A \subset \mathbb{R} \) such that \( \lambda(A) = 0 \) but \( \mu_{SC}(A) = 1 \); here \( \lambda \) is the Lebesgue measure on the real line) and, finally, \( \mu_{AC} \) is absolutely continuous with respect to \( \lambda \).

**Definition.** A distribution of the form (1.5) is called a distribution of type \((\alpha, \beta, \gamma)\). A distribution of type \((1,0,0)\) or \((0,1,0)\) or \((0,0,1)\) is called pure, otherwise it is called a mixture.

A remarkable result of Jessen and Wintner [3] (see [5], page 64, Theorem 3.7.7) is the so called purity theorem (see [2]).

**Purity Theorem.** Let \((X_n)_{n \geq 0}\) be a sequence of independent random variables such that the sequence \((X_0 + \ldots + X_n)_n\) is convergent in distribution to some real random variable \( S \). Then the distribution of \( S \) is pure.

In our one case can say more. The distribution of \( S(q) \) is always continuous (see [5], page 65). Is it absolutely continuous? If \( v \) is absolutely continuous, then it is clear that \( \mu(q) \) is absolutely continuous, too. The reason is that the convolution of \( v \) and any other probability distribution \( \sigma \) is absolutely continuous: if \( g \) is the density of \( v \), then

\[ h(x) = \int g(x-y) d\sigma(y) \]  

is a version for the density of \( v*\sigma \).

If \( v \) is continuous, then it is also easy to see that \( S(q) \) has a continuous distribution, too. For, if \( F \) is the distribution function of \( v \), the distribution function \( G \) of \( v*\sigma \) is given by

\[ G(x) = \int F(x-y) d\sigma(y) \]  

that is, it is continuous, too.

A delicate problem is when \( v \) is discrete. This time is by no means obvious why \( \mu(q) \) should be continuous. It is proved in [5], page 85 that this is indeed the case. The most difficult question is to give a criterion to decide if \( \mu(q) \) is absolutely continuous.

The simplest case is when \( v = \text{Binomial}(1, \frac{1}{2}) \). Now, the distribution of \( S(q) \) is called an infinite Bernoulli convolution (see [2], [3], [4], [5], [7], [8]). It is known that if \( |q| < \frac{1}{2} \) then \( \mu(q) \) is singular (in this case this is almost obvious, since the support of \( \mu(q) \) is negligible), that if \( q = \frac{1}{2} \) then \( \mu(q) = \text{Uniform}(0,2) \), and if \( q \in (\frac{1}{2},1) \setminus M \) then \( \mu(q) \) is absolutely continuous, where \( M \subset (\frac{1}{2}, 1) \) is a negligible set (see [7]). Little is known about the set \( M \). We think that \( M \) is countable. The only \( q \) from \( M \) which is positively known (see [7]) is \( q = \left(\sqrt{5} - 1\right)/2 \), i.e. the solution of the equation \( q + q^2 = 1 \). If \( q \in (-1, -\frac{1}{2}) \), the situation is similar: we can work with the random variables \( Y_n = 2X_n - 1 \) instead of \( X_n \). They are symmetrical, therefore \( aY_n \) and \( -aY_n \) have the same distribution.

Trying to approximate the distribution functions \( F_q \) by \( F_{n,q} \) on the computer we remarked an intriguing regularity of the distribution functions \( F_{n,q} \) : compared with the corresponding uniform distribution function \( G_q(x) = x/L_m \) on \([0, L_m]\) (here \( L_m = 1 + q + \ldots + q^n \) ) they seemed to behave as follows:
- for $q < 1/2 : F_{n,q}(x) > G_{n}(x)$ if $x \in (0, L/2)$ and $F_{n,q}(x) < G_{n}(x)$ if $x \in (L/2, 1)$
- for $q > 1/2 : F_{n,q}(x) < G_{n}(x)$ if $x \in (0, L/2)$ and $F_{n,q}(x) > G_{n}(x)$ if $x \in (L/2, 1)$.

This is remarkable because intersection at one point only of two distribution functions is the Karlin–Novikov criterion for convex domination (see [9] or [10]).

**Definition.** Let $\nu$ and $\sigma$ be two probabilities on the real line. We say that $\nu$ is convex dominated by $\sigma$ and write $\nu \preceq_{\sigma} \sigma$ if \[ \int ud\nu \leq \int ud\sigma \] for all convex functions $u : \mathbb{R} \to \mathbb{R}$ for which the integrals do exist.

If $\mu$ and $\nu$ have the same finite expectation and their distribution functions $F_{\sigma}$ and $F_{\sigma}$ have the property that there exists $x_{0}$ such that $x < x_{0} \Rightarrow F_{\sigma}(x) \leq F_{\sigma}(x)$ and $x \geq x_{0} \Rightarrow F_{\sigma}(x) \geq F_{\sigma}(x)$, then $\nu \preceq_{\sigma} \sigma$. This is the Karlin–Novikov criterion. Unfortunately, it is not equivalent to convex domination.

We intend to prove a weaker result than our empirical remark, namely

**Theorem.** Let $L = 1 + q + q^{2} + \ldots$

If $q < 1/2$ then $\mu(q) \preceq_{\sigma} \text{Uniform}(0,L)$

If $q \in (1/2, 1)$ then $\text{Uniform}(0,L) \prec_{\sigma} \mu(q)$.

**2. A MAJORIZATION LEMMA**

If $A \subset \mathbb{R}$ is a finite set, we shall denote by $U(A)$ the uniform distribution on $A$, precisely

\[ U(A) = \frac{1}{|A|} \sum_{a \in A} \varepsilon_{a}, \]  \hspace{1cm} (2.1)

where $\varepsilon_{a}(B) = 1_{\delta}(a)$ is the Dirac probability at $a$. Notice that if $|A| = |B| = n$, $A = \{a_{0} < a_{1} < \ldots < a_{n}\}$ and $B = \{b_{0} < b_{1} < \ldots < b_{n}\}$, then the definition of convex domination becomes

\[ U(A) \preceq_{\sigma} U(B) \iff u(a_{0}) + u(a_{1}) + \ldots + u(a_{n}) \leq u(b_{0}) + u(b_{1}) + \ldots + u(b_{n}) \]  \hspace{1cm} (2.2)

for any convex function $u$. Letting $u(x) = x$ and $u(x) = -x$ we see that $a_{0} + a_{1} + \ldots + a_{n} = b_{0} + b_{1} + \ldots + b_{n}$. It is well known (and easy to check) that the second inequality is equivalent to

\[ |x - a_{0}| + |x - a_{1}| + \ldots + |x - a_{n}| \leq |x - b_{0}| + |x - b_{1}| + \ldots + |x - b_{n}| \quad \forall x \in \mathbb{R}, \]  \hspace{1cm} (2.3)

It can be proved (see for instance [1] or [6]) that inequality (2.3) is equivalent to

\[ a_{0} \geq b_{0}, a_{0} + a_{1} \geq b_{0} + b_{1}, \ldots, a_{0} + \ldots + a_{n-1} \geq b_{0} + \ldots + b_{n-1}, a_{0} + a_{1} + \ldots + a_{n} = b_{0} + b_{1} + \ldots + b_{n} \]  \hspace{1cm} (2.4)

(Sometimes this is called Karamata’s theorem.) Inequality (2.4) is then written $a \prec b$ (b majorizes a). It is important that in (2.4) we do not need that the numbers $(a_{k})_{n}$ and $(b_{k})_{n}$ be all distinct. A result we need is

**Karamata’s theorem.** Let $a_{0} \leq a_{1} \leq \ldots \leq a_{n}$ and $b_{0} \leq b_{1} \leq \ldots \leq b_{n}$. Let $a = (a_{k})_{n}$ and $b = (b_{k})_{n}$. Then

\[ \sum_{k=0}^{n} \varepsilon_{a_{k}} \prec_{\sigma} \sum_{k=0}^{n} \varepsilon_{b_{k}} \iff a \prec b \]  \hspace{1cm} (2.5)

The proof of our result will rely on

**Lemma 2.1.** Let $q > 0, n \geq 1, \alpha = (n+q)/(2n+1)$. Then

\[ q \in (1/2, n+1) \quad \Rightarrow \quad U(\{0,1,\ldots,n\}) \ast U(\{0,q\}) \prec_{\sigma} U(\{0,\alpha,2\alpha,\ldots,2n+1\}) \]  \hspace{1cm} (2.6)

\[ q \in (0, 1/2) \cup (n+1, \infty) \Rightarrow U(\{0,\alpha,2\alpha,\ldots,2n+1\}) \prec_{\sigma} U(\{0,1,\ldots,n\}) \ast U(\{0,q\}) \]  \hspace{1cm} (2.7)
Notice that \((2n+2) U(\{0,1,\ldots,n\}) * U(\{0,q\}) = \varepsilon_0 + \varepsilon_q + \varepsilon_1 + \varepsilon_{1+q} + \ldots + \varepsilon_n + \varepsilon_{n+q}\) \(\text{(2.8)}\)

Let us arrange ascendingly the numbers \(0,q, 1, 1+q, \ldots, n, n+q\) in the vector \(a = (a_i)_{0 \leq i \leq 2n+1}\) from \(\mathbb{R}^{2n+2}\). Consider also the vector \(b \in \mathbb{R}^{2n+2}\) defined by \(b = (i\alpha)_{0 \leq i \leq 2n+1}. \)

Let \(\Delta_i = (2n+1)(a_0 + a_1 + \ldots + a_i)\) and \(B_i = (2n+1)(b_0 + b_1 + \ldots + b_i)\), \(0 \leq i \leq 2n+1\). Let also \(\Delta_i = A_i - B_i\). Of course \(\Delta_0 = \Delta_{2n+1} = 0\). According to Karamata’s theorem we have to check that

\[ q \in (\frac{1}{2}, n+1) \Rightarrow \Delta_i \geq 0 \quad \forall \, 1 \leq i \leq 2n \quad \text{and} \quad q \in (0, \frac{1}{2}) \cup (n+1, \infty) \Rightarrow \Delta_i \leq 0 \quad \forall \, 1 \leq i \leq 2n \] \(\text{(2.9)}\)

In order to make the computations easier, we shall remark the symmetry

\[ a_{2n+1-i} + a_i = b_{2n+1-i} + b_i = n+q \] \(\text{(2.10)}\)

which further implies the remarkable equality \(\Delta_i = \Delta_{2n-1-i} \forall \, 1 \leq i \leq 2n\). Consequently, it is enough to prove that

\[ q \in (\frac{1}{2}, n+1) \Rightarrow \Delta_i \geq 0 \quad \forall \, 1 \leq i \leq n \quad \text{and} \quad q \in (0, \frac{1}{2}) \cup (n+1, \infty) \Rightarrow \Delta_i \leq 0 \quad \forall \, 1 \leq i \leq n \] \(\text{(2.11)}\)

**Case 1.** The easiest one: \(q \in (0,1)\). Then \((a_i)_{0 \leq i \leq 2n+1} = (0, q, 1+q, 2+q, \ldots, n, n+q)\). It is easy to check that

\[ \Delta_{2i+1} = (2q-1)(i+1)(n-i) \quad \text{and} \quad \Delta_{2i} = (2q-1)(i)(n-i) + 1 \] \(\text{(2.12)}\)

hence (2.9) holds.

**Case 2.** Another easy case: \(q \in [n,\infty)\). Now, \((a_i)_{0 \leq i \leq 2n+1} = (0, 1, 2, \ldots, n, q, 1+q, 2+q, \ldots, n+q)\), and for \(i \leq n\) the reader may check that

\[ 2\Delta_i = i(i+1)(n+1-q), \] \(\text{(2.13)}\)

making obvious claim (2.9).

**Case 3.** \(1 \leq q < n+1\). We have to check that \(\Delta_i \geq 0 \quad \forall \, 1 \leq i \leq n\). Now, we write

\[ n = k + m, q = k + \varepsilon, \quad \text{with} \quad k, m \geq 1 \quad \text{and} \quad 0 \leq \varepsilon < 1. \] \(\text{(2.14)}\)

Notice that \((2n+1)\alpha = 2k + m + \varepsilon \) and \((2n+1)(1-\alpha) = m + 1 - \varepsilon\). This case is more difficult because of the ascending order of the numbers \(i, i+q\) which now becomes \((a_i)_{0 \leq i \leq 2n+1} = (0, 1, 2, \ldots, k, k+\varepsilon, k+1+\varepsilon, \ldots, k+2+\varepsilon, k+m, k+m+\varepsilon, k+m+1+\varepsilon, \ldots, k+m+k+\varepsilon)\).

For \(i \leq n\) the rule is

\[ a_i = i \quad \forall \, 1 \leq i \leq k, \quad a_k = k, \quad a_{k+1} = k + \varepsilon, \ldots, a_{k+i} = k+i, \quad a_{k+i+1} = k + i + \varepsilon, \ldots \] \(\text{(2.15)}\)

Remark that if \(k + 2i < n = k + m\) (hence \(2i < m\) \(\text{then} \)

\[ \delta_i := (2n+1)[(a_{k+i+1} + a_{k+i+2}) - (b_{k+i+1} + b_{k+i+2})] = \frac{(k+1)(k+1-2\varepsilon)(k+1-2\varepsilon)}{2} \] \(\text{(2.16)}\)

(recall that \(k \geq 1 \Rightarrow 2k - 1 + 2\varepsilon \geq 1 + 2\varepsilon\)). On the other hand, as \(\Delta_{k+2i+1} = \Delta_{k+1} + \delta_{0} + \delta_{1} + \ldots + \delta_{i}\), by (2.16), we arrive at

\[ \Delta_{k+2i+1} = \Delta_{k+1} + \delta_{0} + \ldots + \delta_{i} = \frac{k(k+1)}{2}(m+1-\varepsilon) + (2k-1+2\varepsilon)(m-i)(i+1) \] \(\text{(2.17)}\)

making obvious that \(\Delta_{k+2i+1} \geq \Delta_{k+1} \geq 0.\) Moreover, as \(k \geq 1, m \geq 2i \) and \(\varepsilon \geq 0\), we have the inequality

\[ \Delta_{k+2i+1} \geq \frac{k(k+1)}{2}(m+1-\varepsilon) + (2k-1+2\varepsilon)(m-i)(i+1) = \Delta_{k+1} + i^2 + i \] \(\text{(2.18)}\)

Now, write

\[ \Delta_{k+2i} = \Delta_{k+2i-1} + (2n+1)[k+i-(k+2i)\alpha] = \Delta_{k+2i-1} + k(m-2i) + k + i - \varepsilon(k + 2i) \] \(\text{(2.19)}\)

As \(\varepsilon < 1\), we have \(\Delta_{k+2i} \geq \Delta_{k+2i-1} + k(m-2i) + k + i - \varepsilon(k + 2i) \geq \Delta_{k+2i-1} + k(m-2i) - i = \Delta_{k+2i-1} + i \). By (2.18), we see that \(\Delta_{k+2i} \geq \Delta_{k+1} + i\). Consequently, \(\Delta \geq \Delta_{k+1} > 0 \quad \forall \, t = k, k+1, \ldots, n\). This completes the proof.

Actually we shall use an obvious generalization of Lemma 2.1, namely...
Corollary 2.2. Let $N \geq 1$, $\delta$, $r > 0$ and $\alpha = \delta(N+r)/(2N+1)$. Then

$$r \in (\delta/2, N+1) \Rightarrow U(\{0, \delta, \ldots, N\delta\}) * U(\{0, r\delta\}) \prec_{\alpha} U(\{0, \alpha, 2\alpha, \ldots, (2N+1)\alpha\}) \quad (2.19)$$

and

$$q \in (0, \delta/2) \cup (N+1, \infty) \Rightarrow U(\{0, \alpha, 2\alpha, \ldots, (2N+1)\alpha\}) \prec_{\alpha} U(\{0, \delta, \ldots, N\delta\}) * U(\{0, r\delta\}). \quad (2.20)$$

3. THE PROOF OF THE THEOREM

Clearly, the distribution $\mu(n,q)$ can be written as

$$\mu(n,q) = U(\{0,1\}) * U(\{0,q\}) * \ldots * U(\{0,q^n\}) \quad (3.1)$$

Suppose that $q > \delta/2$. According to Lemma 2.1, $\mu(2,q) \prec_{\alpha} U(\{0, \delta, 2\delta, 3\delta\})$ where $3\delta = 1 + q$. Now, we want to apply Corollary 2.2. with $r\delta = q^2$. In order to do that, we should check that $\delta/2 \leq r \leq 3 + 1 \Rightarrow \delta/2 \leq q^2/\delta \leq 4 \Leftrightarrow \delta/2 \leq 3q^2/(1+q^2) \leq 4$ or, in other words, that $1 + q \leq 6q^2 \leq 8$. As $\delta/2 < q < 1$, this is obvious. Thus, applying the monotonicity property of the convex domination (i.e. $\mu \prec_{\alpha} \nu \Rightarrow \mu^* \prec_{\alpha} \nu^*$, see for instance [8], [9]) we get $\mu(3,q) = \mu(2,q) * U(\{0,q^2\}) \prec_{\alpha} U(\{0, \delta, 2\delta, 3\delta\}) * U(\{0,q^2\})$ and

$$\mu(n-1,q) \prec_{\alpha} U(\{0, \delta, 2\delta, \ldots, (2^n-1)\delta\}) \quad (2.20)$$

Next, we know that $\mu(n,q) = \mu(n-1,q) * U(\{0,q^n\}) \prec_{\alpha} U(\{0, \delta, 2\delta, \ldots, (2^n-1)\delta\}) * U(\{0,q^n\})$. In order to apply Corollary 2.2, we check that $\delta/2 \leq q^2/\delta \leq 2^n - 1 + 1$ or, explicitly, that

$$\frac{1}{2} \leq q^2(1 + \frac{1}{q} + \frac{1}{q^2} + \ldots + \frac{1}{q^{n-1}}) \leq 2^n \quad (3.2)$$

As $1/q < 2$, we have

$$q^2(1 + \frac{1}{q} + \frac{1}{q^2} + \ldots + \frac{1}{q^{n-1}}) \geq q^2(1 + \frac{1}{q} + \frac{1}{q^2} + \ldots + \frac{1}{q^{n-1}}) \geq 1 + 2 + 2^2 + \ldots + 2^{n-1}$$

hence the left inequality is clear. We have to prove the right one, which can be written as

$$q^n(2^n - 1) \leq 2^n$$

or

$$(2^n - 1)(q^n - q^{n+1}) \leq 2^n(1 - q^n) \quad (3.3)$$

But the function $f(q) = (2^n - 1)(q^n - q^{n+1}) - 2^n(1 - q^n)$ has the properties: $f(0) = -2^n$, $f(1) = 0$, and is increasing on the interval $[0,1]$, thus it is negative. It means that $U(\{0, \delta, 2\delta, \ldots, (2^n-1)\delta\}) \prec_{\alpha} U(\{0, \alpha_n, 2\alpha_n, \ldots, (2^{n+1}-1)\alpha_n\})$ with $(2^{n+1}-1)\alpha_n = 1 + q + \ldots + q^n$. Consequently, we proved the domination $\mu(n,q) \prec_{\alpha} U(\{0, \delta, 2\delta, \ldots, (2^{n+1}-1)\delta\})$ for any $n \geq 1$ where $(2^{n+1}-1)\delta = 1 + q + \ldots + q^n$.

If $q < \delta/2$, then $1/q > 2$ hence
By Corollary 2.2 the domination goes into the opposite direction.

The rest of the proof is routine: \( \mu(n,q) \) converges to \( \mu(q) \), \( U(\{0, \alpha_n, 2\alpha_n, \ldots, (2^n+1)\alpha_n\}) \) converges to Uniform(0, \( L \)) with \( 1/L = 1 - q \) and the convergence is dominated, in the sense that the supports of all these measures are included in [0,\( L \)]. But it is well known – and easy to check – that if \( \mu_n \Rightarrow \mu, \ \nu_n \Rightarrow \nu, \ Supp(\mu_n) \cup Supp(\nu_n) \subset K, \ K \) compact, then \( \mu \prec_\text{ex} \nu. \)

**Corollary 3.2** (Moments and moment generating function). Let \( q \in (\frac{1}{2}, 1) \), \( n \geq 2, \ t \geq 0 \) and \( 1/L = 1 - q \).

Then

\[
E S^n (q) \leq \frac{1}{(n+1)(1-q)^n} \quad \text{and} \quad E e^{tS(q)} \leq \frac{e^{tL} - 1}{tL}
\]

**Proof.** The functions \( x \mapsto x^n \) ant \( x \mapsto e^{tx} \), \( x \geq 0 \), are convex and the distribution of \( S(q) \) is dominated by the uniform one. The second inequality can also be written as

\[
\lim_{n \to \infty} \frac{e^t - 1}{t} e^{q^n t} - 1 = \frac{e^{tL} - 1}{tL}
\]

If \( q = \frac{1}{2} \) (thus \( L = 2 \)) we get a strange equality.

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**REFERENCES**


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