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ON INFINTE BERNOULLI CONVOLUTIONS

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Let $(X_n)_{n\geq 0}$ be a sequence of i.i.d. nondegenerate integrable random variables and let $q \in [0,1]$. Let $S(n,q) = X_0 + qX_1 + \ldots + q^nX_n$. If $|q| < 1$ the sequence $(S(n,q))_{n\geq 0}$ is almost surely convergent to a integrable random variable $S(q)$ which has a distribution denoted by $\mu(q)$. Even in the most simple case when $X_n \sim$ Binomial(1, ½) behaves mysteriously erratic when $q \in$ (½ ,1). We prove that there still exists a regularity, namely

$$0 < q < ½ \Rightarrow \text{Uniform}(0,L) \prec_{cx} \mu(q)$$

and

$$½ < q < 1 \Rightarrow \mu(q) \prec_{cx} \text{Uniform}(0,L),$$

where $1/L = 1 - q$ and "$\prec_{cx}$" is the Choquet convex domination. The problem has a clear financiary motivation: if q is an actualization factor, then $S(q)$ is the actual value of the infinite sum $X_0 + X_1 + \ldots$.

1. THE PROBLEM

Let $(X_n)_{n \geq 0}$ be a sequence of i.i.d. random variables and $S(n,q) = X_0 + qX_1 + \ldots + q^nX_n$, with q a real number. As

$$S(n+k,q) - S(n,q) \sim q^{n+1} S(k-1,q) \tag{1.1}$$

(the notation X ~ Y means that X and Y have the same distribution), it is obvious that the sequence $(S(n,q))_{n \geq 0}$ diverges for any $q \in (-\infty, -1] \cup [1, \infty)$.
What does happen if $q \in (-1,1)$?
If the random variables X_n are not integrable, it is possible that the sequence $(S(n,q))_{n \geq 0}$ diverge. However, if they are integrable, that is not possible since

$$\|S(n+k,q) - S(n,q)\|_1 \leq |q|^{n+1} \sum_{j\geq 0} |q|^j E|X_n| = \frac{|q|^{n+1}}{1-|q|} \|X_1\|_1 \tag{1.2}$$

meaning that $(S(n,q))_{n \geq 0}$ is Cauchy in L^1, hence convergent in L^1. It is easy to check that it is also convergent a.s. since the series $S(q) = \sum_{n=0}^{\infty} q^n X_n$ converges a.s. If $X_n \in L^\infty$, the convergence is even uniform.

The real problem is to compute the distribution of $S(q)$. Let F_q and $F_{n,q}$ be the distribution functions of $S(q)$ and $S(n,q)$. Let also ν be the distribution of X_n and $\mu(q)$, $\mu(n,q)$ the distributions of $S(q)$ and $S(n,q)$.

If $|X_n| \leq M$ a.s. (that is, X_n are essentially bounded), it is easy to see that

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$$S(n,q) - \frac{|q|^{n+1}M}{1-|q|} \le S(q) \le S(n,q) + \frac{|q|^{n+1}M}{1-|q|} \tag{1.3}$$

Therefore, a coarse evaluation of F_q would be

$$F_{n,q}(x - \frac{|q|^{n+1}M}{1-|q|}) \le F_q(x) \le F_{n,q}(x + \frac{|q|^{n+1}M}{1-|q|}) \tag{1.4}$$

which, for great *n,* is good enough for continuity points of $F_{n,q}$.

Anyway, estimation (1.4) is useless if we want to know the *type* of the distribution $\mu(q)$. According to the Lebesgue – Nikodym theorem any probability distribution μ on the real line can be written as a mixture

$$\mu = \alpha\mu_D + \beta\mu_{SC} + \gamma\mu_{AC,} \tag{1.5}$$

where $\alpha,\beta,\gamma \ge 0$, $\alpha + \beta + \gamma + 1$, μ_D is a discrete distribution, μ_{SC} is continuous but singular (i.e. there exists a Borel set $A \subset \Re$ such that $\lambda(A) = 0$ but $\mu_{SC}(A) = 1$; here λ is the Lebesgue measure on the real line) and, finally, μ_{AC} is absolutely continuous with respect to λ.

Definition. A distribution of the form (1.5) is called a distribution of type (α,β,γ). A distribution of type (1,0,0) or (0,1,0) or (0,0,1) is called **pure** , otherwise it is called a **mixture.**

A remarkable result of Jessen and Wintner [3] (see [5], page 64, Theorem 3.7.7) is the so called **purity theorem** (see [2]).

Purity Theorem . *Let $(X_n)_{n\ge 0}$ be a sequence of independent random variables such that the sequence $(X_0 + \ldots + X_n)_n$ is convergent in distribution to some real random variable S. Then the distribution of S is pure.*

In our one case can say more. The distribution of $S(q)$ is always continuous (see [5], page 65). Is it absolutely continuous? If ν is absolutely continuous, then it is clear that $\mu(q)$ is absolutely continuous, too. The reason is that the convolution of ν and any other probability distribution σ is absolutely continuous: if g is the density of ν, then

$$h(x) = \int g(x-y)\mathrm{d}\sigma(y) \tag{1.6}$$

is a version for the density of $\nu*\sigma$.

If ν is continuous, then it is also easy to see that $S(q)$ has a continuous distribution, too. For, if F is the distribution function of ν, the distribution function G of $\nu*\sigma$ is given by

$$G(x) = \int F(x-y)\mathrm{d}\sigma(y), \tag{1.7}$$

that is, it is continuous, too.

A delicate problem is when ν is discrete. This time is by no means obvious why $\mu(q)$ should be continuous. It is proved in [5], page 85 that this is indeed the case. The most difficult question is to give a criterion to decide if $\mu(q)$ is absolutely continuous.

The simplest case is when ν = Binomial(1, ½). Now, the distribution of $S(q)$ is called an ***infinite Bernoulli convolution*** (see [2], [3], [4], [5], [7], [8]). It is known that if $|q| < ½$ then $\mu(q)$ is singular (in this case this is almost obvious, since the support of $\mu(q)$ is negligible), that if $q = ½$ then $\mu(q)$ = Uniform(0,2), and if $q \in (½,1) \setminus M$ then $\mu(q)$ is absolutely continuous, where $M \subset (½, 1)$ is a negligible set (see [7]). Little is known about the set M. We think that M is countable. The only q from M which is positively known (see [7]) is $q = (\sqrt{5}-1)/2$, i.e. the solution of the equation $q + q^2 = 1$. If $q \in (-1, -½)$, the situation is similar: we can work with the random variables $Y_n = 2X_n - 1$ instead of X_n. They are symmetrical, therefore aY_n and $-aY_n$ have the same distribution.

Trying to approximate the distribution functions F_q by $F_{n,q}$ on the computer we remarked an intriguing regularity of the distribution functions $F_{n,q}$: compared with the corresponding uniform distribution function $G_n(x) = x/L_n$ on $[0, L_n]$ (here $L_n = 1 + q + \ldots + q^n]$) they seemed to behave as follows:

- for $q < ½$: $F_{n,q}(x) > G_n(x)$ if $x \in (0, L_n/2)$ and $F_{n,q}(x) < G_n(x)$ if $x \in (L_n/2, 1)$
- for $q > ½$: $F_{n,q}(x) < G_n(x)$ if $x \in (0, L_n/2)$ and $F_{n,q}(x) > G_n(x)$ if $x \in (L_n/2, 1)$.

This is remarkable because intersection at one point only of two distribution functions is the Karlin – Novikov criterion for ***convex domination*** (see [9] or [10]).

Definition. *Let ν and σ be two probabilities on the real line. We say that ν is **convex dominated** by σ–and write $\nu \prec_{\mathbf{cx}} \sigma$ if $\int u d\nu \le \int u d\sigma$ for all convex functions $u : \Re \to \Re$ for which the integrals do exist.*

If μ and ν have the same finite expectation and their distribution functions F_ν and F_σ have the property that there exists x_0 such that $x < x_0 \Rightarrow F_\nu(x) \le F_\sigma(x)$ and $x \ge x_0 \Rightarrow F_\nu(x) \ge F_\sigma(x)$, then $\nu \prec_{\mathbf{cx}} \sigma$. This is the Karlin – Novikov criterion. Unfortunatel, it is not equivalent to convex domination.

We intend to prove a weaker result than our empirical remark, namely

Theorem. *Let $L = 1 + q + q^2 + \ldots$.*

If $q < ½$ then $\mu(q) \prec_{\mathbf{cx}}$ Uniform$(0,L)$

If $q \in (½, 1)$ then Uniform$(0,L) \prec_{\mathbf{cx}} \mu(q)$.

2. A MAJORIZATION LEMMA

If $A \subset \Re$ is a finite set, we shall denote by $U(A)$ the uniform distribution on A, precisely

$$U(A) = \frac{1}{|A|}\sum_{a\in A}\varepsilon_a , \tag{2.1}$$

where $\varepsilon_a(B) = 1_B(a)$ is the Dirac probability at a. Notice that if $|A| = |B| = n$, $A = \{a_0 < a_1 <\ldots< a_n\}$ and $B = \{b_0 < b_1 < \ldots < b_n\}$, then the definition of convex domination becomes

$$U(A) \prec_{\mathbf{cx}} U(B) \quad \Leftrightarrow \; u(a_0) + u(a_1) + \ldots + u(a_n) \le u(b_0) + u(b_1) + \ldots + u(b_n) \tag{2.2}$$

for any convex function u. Letting $u(x) = x$ and $u(x) = -x$ we see that $a_0 + a_1 + \ldots + a_n = b_0 + b_1 + \ldots + b_n$.
It is well known (and easy to check) that the second inequality is equivalent to

$$|x - a_0| + |x - a_1| + \ldots + |x - a_n| \le |x - b_0| + |x - b_1| + \ldots + |x - b_n| \;\; \forall x \in \Re, \tag{2.3}$$

It can be proved (see for instance [1] or [6]) that inequality (2.3) is equivalent to

$$a_0 \ge b_0,\ a_0 + a_1 \ge b_0 + b_1, \ldots,\ a_0 + \ldots + a_{n-1} \ge b_0 + \ldots + b_{n-1},\ a_0 + a_1 + \ldots + a_n = b_0 + b_1 + \ldots + b_n \tag{2.4}$$

(Sometimes this is called Karamata's theorem.) Inequality (2.4) is then written $a \prec b$ (b majorizes a). It is important that in (2.4) we do not need that the numbers $(a_k)_k$ and $(b_k)_k$ be all distinct. A result we need is

Karamata's theorem. *Let $a_0 \le a_1 \le \ldots \le a_n$ and $b_0 \le b_1 \le \ldots \le b_n$. Let $a = (a_k)_k$ and $b = (a_k)_k$. Then*

$$\sum_{k=0}^{n}\varepsilon_{a_i} \prec_{\mathbf{cx}} \sum_{k=0}^{n}\varepsilon_{a_i} \Leftrightarrow a \prec b \tag{2.5}$$

The proof of our result will rely on

Lemma 2.1. *Let $q > 0$, $n \ge 1$, $\alpha = (n+q)/(2n+1)$. Then*

$$q \in (½, n+1) \quad\quad \Rightarrow U(\{0,1,\ldots,n\}) * U(\{0,q\}) \prec_{\mathbf{cx}} U(\{0,\alpha,2\alpha,\ldots,(2n+1)\alpha\}) \tag{2.6}$$

$$q \in (0, ½) \cup (n+1, \infty) \Rightarrow U(\{0,\alpha,2\alpha,\ldots,(2n+1)\alpha\}) \prec_{\mathbf{cx}} U(\{0,1,\ldots,n\}) * U(\{0,q\}) . \tag{2.7}$$

Proof. Notice that

$$(2n+2)\, U(\{0,1,\ldots,n\}) * U(\{0,q\}) = \varepsilon_0 + \varepsilon_q + \varepsilon_1 + \varepsilon_{1+q} + \ldots + \varepsilon_n + \varepsilon_{n+q} \tag{2.8}$$

Let us arrange ascendingly the numbers $0,q,\ 1,\ 1+q,\ \ldots,n,n+q$ in the vector $a = (a_i)_{0\le i\le 2n+1}$ from $\Re^{2n+2}$. Consider also the vector $b \in \Re^{2n+2}$ defined by $b = (i\alpha)_{0\le i\le 2n+1}$. Let $A_i = (2n+1)(a_0 + a_1 + \ldots + a_i)$ and $B_i = (2n+1)(b_0 + b_1 + \ldots + b_i)$, $0 \le i \le 2n+1$. Let also $\Delta_i = A_i - B_i$. Of course $\Delta_0 = \Delta_{2n+1} = 0$. According to Karamata's theorem we have to check that

$$q \in (\tfrac{1}{2}, n+1) \Rightarrow \Delta_i \ge 0\ \forall\ 1 \le i \le 2n \text{ and } q \in (0,\tfrac{1}{2}) \cup (n+1,\infty) \Rightarrow \Delta_i \le 0\ \forall\ 1 \le i \le 2n \tag{2.9}$$

In order to make the computations easier, we shall remark the symmetry

$$a_{2n+1-i} + a_i = b_{2n+1-i} + b_i = n+q \tag{2.10}$$

which further implies the remarkable equality $\Delta_i = \Delta_{2n-i}\ \forall\ 1 \le i \le 2n$. Consequently, it is enough to prove that

$$q \in (\tfrac{1}{2}, n+1) \Rightarrow \Delta_i \ge 0\ \forall\ 1 \le i \le n \text{ and } q \in (0,\tfrac{1}{2}) \cup (n+1,\infty) \Rightarrow \Delta_i \le 0\ \forall\ 1 \le i \le n \tag{2.11}$$

Case 1. The easiest one: $q \in (0,1]$. Then $(a_i)_{0\le i\le 2n+1} = (0, q, 1, 1+q, 2, 2+q, \ldots, n, n+q)$. It is easy to check that

$$\Delta_{2i+1} = (2q-1)(i+1)(n-i) \text{ and } \Delta_{2i} = (2q-1)[(i+1)(n-i) + i] \tag{2.12}$$

hence (2.9) holds.

Case 2. Another easy case: $q \in [n,\infty)$. Now, $(a_i)_{0\le i\le 2n+1} = (0, 1,2,\ldots, n, q,1+q, 2+q, \ldots, n+q)$, and for $i \le n$ the reader may check that

$$2\Delta_i = i(i+1)(n+1-q), \tag{2.13}$$

making obvious claim (2.9).

Case 3. $1 \le q < n+1$. We have to check that $\Delta_i \ge 0\ \forall\ 1 \le i \le n$. Now, we write

$$n = k + m,\ q = k + \varepsilon, \text{ with } k,m \ge 1 \text{ and } 0 \le \varepsilon < 1. \tag{2.14}$$

Notice that $(2n+1)\alpha = 2k + m + \varepsilon$ and $(2n+1)(1-\alpha) = m + 1 - \varepsilon$. This case is more difficult because of the ascending order of the numbers $i,i+q$ which now becomes
$(a_i)_{0\le i\le 2n+1} = (0, 1, 2,\ldots, k, k+\varepsilon, k+1, k+1+\varepsilon, k+2, k+2+\varepsilon, k+m, k+m+\varepsilon, k+m+1+\varepsilon,\ldots, k+m+k+\varepsilon)$.
For $i \le n = k+m$ the rule is

$$a_i = i\ \forall\ 1 \le i \le k,\ a_k = k,\ a_{k+1} = k + \varepsilon,\ldots,a_{k+2i} = k+i,\ a_{k+2i+1} = k + i + \varepsilon,\ldots \tag{2.15}$$

Remark that if $k + 2i < n = k + m$ (hence $2i < m$) then

$$\delta_i := (2n+1)[(a_{k+2i} + a_{k+2i+1}) - (b_{k+2i} + b_{k+2i+1})] = (m-2i)(2k-1+2\varepsilon) > 0 \tag{2.16}$$

(recall that $k \ge 1 \Rightarrow 2k - 1 + 2\varepsilon \ge 1 + 2\varepsilon$!). On the other hand, as $\Delta_{k+2i+1} = \Delta_{k-1} + \delta_0 + \delta_1 + \ldots + \delta_i$, by (2.16) we arrive at

$$\Delta_{k+2i+1} = \Delta_{k-1} + (\delta_0 + \ldots + \delta_i) = \frac{k(k-1)}{2}(m+1-\varepsilon) + (2k-1+2\varepsilon)(m-i)(i+1) \tag{2.17}$$

making obvious that $\Delta_{k+2i+1} \ge \Delta_{k-1} \ge 0$. Moreover, as $k \ge 1$, $m \ge 2i$ and $\varepsilon \ge 0$, we have the inequality

$$\Delta_{k+2i+1} \ge \frac{k(k-1)}{2}(m+1-\varepsilon) + (2\cdot 1 - 1)(2i-i)(i+1) = \Delta_{k-1} + i^2 + i \tag{2.18}$$

Now, write
$\Delta_{k+2i} = \Delta_{k+2i-1} + (2n+1)[k + i - (k+2i)\alpha] = \Delta_{k+2i-1} + k(m-2i) + k + i - \varepsilon(k+2i)$. As $\varepsilon < 1$, we have $\Delta_{k+2i} \ge \Delta_{k+2i-1} + k(m-2i) + k + i - (k+2i) = \Delta_{k+2i-1} + k(m-2i) - i = \Delta_{k+2i+1} - i$. By (2.18), we see that $\Delta_{k+2i} \ge \Delta_{k-1} + i^2$. Consequently, $\Delta_t \ge \Delta_{k-1} > 0\ \forall\ t = k,k+1,\ldots, n$. This completes the proof.

Actually we shall use an obvious generalization of Lemma 2.1, namely

Corollary 2.2. *Let* $N \geq 1$, $\delta, r > 0$ *and* $\alpha = \delta(N+r)/(2N+1)$. *Then*

$$r \in (½, N+1) \Rightarrow U(\{0,\delta,\ldots,N\delta\}) * U(\{0,r\delta\}) \prec_{\mathbf{cx}} U(\{0,\alpha,2\alpha,\ldots,(2N+1)\alpha\}) \tag{2.19}$$

and

$$q \in (0, ½) \cup (N+1, \infty) \Rightarrow U(\{0,\alpha,2\alpha,\ldots,(2N+1)\alpha\}) \prec_{\mathbf{cx}} U(\{0,\delta,\ldots,N\delta\}) * U(\{0,r\delta\}) . \tag{2.20}$$

3. THE PROOF OF THE THEOREM

Clearly, the distribution $\mu(n,q)$ can be written as

$$\mu(n,q) = U(\{0,1\})*U(\{0,q\})*\ldots*U(\{0,q^n\}) \tag{3.1}$$

Suppose that $q > ½$. According to Lemma 2.1, $\mu(2,q) \prec_{\mathbf{cx}} U(\{0,\delta, 2\delta, 3\delta\})$ where $3\delta = 1 + q$. Now, we want to apply Corollary 2.2. with $r\delta = q^2$. In order to do that, we should check that $½ \leq r \leq 3+1 \Leftrightarrow ½ \leq q^2/\delta \leq 4 \Leftrightarrow ½ \leq 3q^2/(1+q) \leq 4$ or, in other words, that $1 + q \leq 6q^2 \leq 8$. As $½ < q < 1$, this is obvious. Thus, applying the monotonicity property of the convex domination (i.e. $\mu \prec_{\mathbf{cx}} \nu$, $\mu' \prec_{\mathbf{cx}} \nu' \Rightarrow \mu*\mu' \prec_{\mathbf{cx}} \nu*\nu'$, see for instance [8], [9]) we get $\mu(3,q) = \mu(2,q)*U(\{0,q^2\}) \prec_{\mathbf{cx}} U(\{0,\delta, 2\delta, 3\delta\})* U(\{0,q^2\}) \prec U(\{0,\alpha,2\alpha,\ldots,7\alpha\})$ with $\alpha =(1+q+q^2) / 7$.

Suppose that we proved that $\mu(n\text{-}1,q) \prec_{\mathbf{cx}} U(\{0, \delta,2\delta,\ldots, (2^n\text{-}1)\delta\})$ where $(2^n\text{-}1)\delta = 1 + q+\ldots+ q^{n-1}$. Next, we know that $\mu(n,q) = \mu(n\text{-}1,q)*U(\{0,q^n\}) \prec_{\mathbf{cx}} U(\{0, \delta,2\delta,\ldots, (2^n\text{-}1)\delta\})*U(\{0,q^n\})$. In order to apply Corollary 2.2, we check that $½ \leq q^n/ \delta \leq 2^n - 1 + 1$ or, explicitely, that

$$\frac{1}{2} \leq q\frac{1+2+2^2+\ldots+2^{n-1}}{1+\frac{1}{q}+\left(\frac{1}{q}\right)^2+\ldots+\left(\frac{1}{q}\right)^{n-1}} \leq 2^n \tag{3.2}$$

As $1 / q < 2$, we have

$$q\frac{1+2+2^2+\ldots+2^{n-1}}{1+\frac{1}{q}+\left(\frac{1}{q}\right)^2+\ldots+\left(\frac{1}{q}\right)^{n-1}} \geq q\frac{1+2+2^2+\ldots+2^{n-1}}{1+2+2^2+\ldots+2^{n-1}}$$

hence the left inequality is clear. We have to prove the right one, which can be written as

$$\frac{q^n(2^n-1)}{1+q+q^2+\ldots+q^{n-1}} \leq 2^n$$

or

$$(2^n - 1)(q^n - q^{n+1}) \leq 2^n(1 - q^n) \ \ \forall\, q \in (0,1). \tag{3.3}$$

But the function $f(q) = (2^n - 1)(q^n - q^{n+1}) - 2^n(1 - q^n)$ has the properties: $f(0) = -2^n$, $f(1) = 0$, and is increasing on the interval [0,1], thus it is negative. It means that $U(\{0, \delta,2\delta,\ldots, (2^n\text{-}1)\delta\})*U(\{0,q^n\}) \prec_{\mathbf{cx}} U(\{0, \alpha_n,2\alpha_n,\ldots, (2^{n+1}\text{-}1)\alpha_n\})$ with $(2^{n+1}\text{-}1)\alpha_n = 1 + q + \ldots+ q^n$. Consequenly, we proved the domination $\mu(n,q) \prec_{\mathbf{cx}} U(\{0, \delta,2\delta,\ldots, (2^{n+1}\text{-}1)\delta\})$ for any $n \geq 1$ where $(2^{n+1}\text{-}1)\delta = 1 + q+\ldots+ q^n$.

If $q < ½$, then $1 / q > 2$ hence

$$q\frac{1+2+2^2+\ldots+2^{n-1}}{1+\frac{1}{q}+\left(\frac{1}{q}\right)^2+\ldots+\left(\frac{1}{q}\right)^{n-1}}\le q\frac{1+2+2^2+\ldots+2^{n-1}}{1+2+2^2+\ldots+2^{n-1}}$$

By Corollary 2.2 the domination goes into the opposite direction.

The rest of the proof is routine: $\mu(n,q)$ converges to $\mu(q)$, $U(\{0, \alpha_n, 2\alpha_n, \ldots, (2^{n+1}-1)\alpha_n\})$ converges to Uniform(0, L) with $1/L = 1 - q$ and the convergence is dominated, in the sense that the supports of all these measureas is included in $[0,L]$. But it is well known – and easy to check – that if

$\mu_n \Rightarrow \mu$, $\nu_n \Rightarrow \nu$, $\mu_n \prec_{\mathbf{cx}} \nu_n$ $\mathrm{Supp}(\mu_n) \cup \mathrm{Supp}(\nu_n) \subset K$, K compact, then $\mu \prec_{\mathbf{cx}} \nu$.

Corollary 3.2 (Moments and moment generatig function). *Let* $q \in (\frac{1}{2}, 1)$, $n \ge 2$, $t \ge 0$ *and* $1/L = 1 - q$. *Then*

$$\mathrm{E}S^n(q) \le \frac{1}{(n+1)(1-q)^n} \quad \textit{and} \quad \mathrm{E}e^{tS(q)} \le \frac{e^{tL}-1}{tL}$$

Proof. The functions $x \mapsto x^n$ ant $x \mapsto e^{tx}$, $x \ge 0$, are convex and the distribution of $S(q)$ is dominated by the uniform one. The second inequality can also be written as

$$\lim_{n\to\infty}\frac{e^t-1}{2}\frac{e^{qt}-1}{2}\cdots\cdots\frac{e^{q^n t}-1}{2}\le\frac{e^{tL}-1}{tL}$$

If $q = \frac{1}{2}$ (thus $L = 2$) we get a strange equality.

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REFERENCES

1. ARNOLD, B., *Majorization and the Lorenz Order: a Brief Introduction,* Lecture Notes in Statistics **43**, Berlin, Springer, 1987.
2. DIACONIS, P., FREEDMAN, D., *Iterated random function.,* SIAM Review, **41**, 1, pp. 45-76, 1999.
3. JESSEN, W., WINTNER, A., *Distribution functions and the Riemann Zeta function,* Trans. Amer. Math. Soc. **38**, pp. 48 – 88, 1935.
4. HERKENRATH, U., IOSIFESCU, M., *On positive Harris recurrence for a class of Markov chains,* Math. Reports **7** (**57**), 4 , pp. 305 – 313, 2005.
5. LUKACS, E., *Characteristic Functions.* London, Griffin &Co, 1970.
6. MARSHALL, A. W., OLKIN, I., *Inequalities: Theory of Majorization and its Applications,* New York, Academic Press, 1979.
7. PERES, Y., SOLOMYAK, B., *Absolute continuity of Bernoulli convolutions, a simple proof,* Math. Research Letters **3**, pp. 231 – 239, 1996.
8. PERES, Y., SCHLAG, W.,SOLOMYAK, B., *Sixty years of Berrnoulli convolutions.* In: *Fractal Geometry and Stochastic II* (*Greifswald/Keserow* 1998), pp. 39-45*,* Progr. Probab., **46**, Basel, Birkhauser, 2000.
9. SZEKLI, R., *Stochastic Ordering and Dependence in Applied Probability,* Berlin, Springer, 1995.
10. ZBĂGANU, G., *Mathematical Methods in Risk Theory and Actuaries.* Bucharest, Ed. Univ. Bucharest, 2004. (Romanian)

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