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INVARIANT PROPERTIES OF QUOTIENT GROUPS IN COMMUTATIVE MODULAR GROUP ALGEBRAS

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Suppose G is an abelian group with p-component G_p and R is a perfect commutative unitary ring without zero divisors of prime characteristic, for instance, p. It is shown that if the group algebras RG and RH are R-isomorphic for any group H, then some principal properties of the Sylow p-subgroup $S(RG)$ modulo G_p, that is $S(RG)/G_p$, are saved for $S(RH)/H_p$ too. In particular, as a subsequent application, more smooth proofs of well-known results are given as well.

Key words: factor-groups, isomorphisms, direct factors, unit groups, abelian groups.

I. INTRODUCTION

Throughout the text, let RG be the group algebra of an abelian multiplicatively written group G over a commutative ring R with identity and prime characteristic p. For any subgroup C of G, we define $I(RG; C)$ as the relative augmentation ideal of RG with respect to C. We denote by $S(RG)$ the group of all normalized p-elements in RG, and by G_p the group of all p-elements in G. Since there is a natural embedding $G \subset RG$, we have $G_p \subseteq S(RG)$. Thus we form the factor-group $S(RG)/G_p$, which is in the focus of our interest. We feel that a problem of some importance, which perhaps has a connection with the long-standing and very difficult *Isomorphism Problem*, is the following one.

Problem: Assume $RG \cong RH$ as R-algebras. If

1. $S(RG)/Gp$ possesses the property $\wp$ (in particular $S(RG)/Gp \in \Re$, a class of abelian p-groups), whether the same does hold valid for $S(RH)/Hp$ and, even more, whether $S(RG)/Gp \cong S(RH)/Hp$;
2. Gp is a direct factor of $S(RG)$, whether Hp is a direct factor of $S(RH)$.

Comments.

1. Inspired by the *Generalized Direct Factor Conjecture* (= GDFC), which says that $S(RG)/G_p$ is always totally projective whenever G_p is reduced and R is perfect, we claim that the isomorphism $S(RG)/G_p \cong S(RH)/H_p$ is ever fulfilled provided, of course, $RG \cong RH$.

   In fact, if the GDFC holds in the affirmative, then $S(RG)/G_p \cong S(RH)/H_p$ follows with the help of [13], or with the aid of the Theorem proved below in the case when $G = G_p$, or utilizing the Proposition argued in the sequel jointly with [18].
2. Here, we follow two independent approaches.

First of all, utilizing again the truthfulness of the GDFC, both $S(RG)/G_p$ and $S(RH)/H_p$ should be totally projective. Henceforth, invoking to the simple fact (cf., e.g., [3], [4], [9], [11], [13], [16]) that G_p and H_p are balanced (= nice + isotype) subgroups of $S(RG)$ and $S(RH)$, respectively, one can deduce at once that the isomorphisms $S(RG) \cong G_p \times S(RG)/G_p$ and $S(RH) \cong H_p \times S(RH)/H_p$ are always possible.

Secondly, we will use the *Generalized Isomorphism Conjecture* (= GIC), which states that $RG \cong RH$ implies $G_p \cong H_p$. We know that $RG \cong RH$ forces that S$(RG) \cong S(RH)$. Thus, if the answer of this problem is positive, in view of the isomorphism S$(RG)/G_p \cong S(RH)/H_p$ we can write $S(RG) \cong G_p \times S(RG)/G_p \cong H_p \times$

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$S(RH)/H_p \cong S(RH)$. Thereby, the last isomorphism plainly leads us by [2] to H_p is a direct factor of $S(RH)$, as desired.

Because of technical difficulties, we shall be more concrete only in the situation when $H = H_p$, i.e. when H is a p-group. In that variant a technicality due to May ([17], Lemma 2) works. In fact, $S(RH) \cong H \times S(RH)/H$ means that H is isomorphic to a direct factor of $S(RH)$, say D. We now provide a more circumstantial confirmation that H must be a direct factor of $S(RH)$. In order to show this, consider the isomorphism map $f: H \to D$. Since $D \subseteq S(RH)$, we traditionally extend it to the homomorphism $f': S(RH) \to S(RH)$ such that $f'_H = f$. Moreover, we consider the projection $\pi: S(RH) \to D$ that is the identity map on D, namely $\pi(d) = d$, $\forall d \in D$. Consequently, $f^{-1}\pi f': S(RH) \to H$ is a homomorphism so that for each $h \in H$ we have $f^{-1}\pi f'(h) = f^{-1}\pi(d) = f^{-1}(d) = h$, because $f'(h) = f(h) = d$. Therefore, $f^{-1}\pi f'$ is a projection and thus H is really a direct factor of $S(RH)$, proving the claim.

The purpose of the present research article is to resolve (in a slightly modified form) the first of the foregoing two posed questions in some special cases. As a consequence, we shall obtain easy proofs of our statements (see, for example, [4], [5], [9], [12]) concerning the isomorphism conjecture. It is worthwhile noticing that a part of the assertions quoted below was preliminary announced in [6, section III].

2. THE MAIN ATTAINMENT ON INVARIANCES IN COMMUTATIVE MODULAR GROUP ALGEBRAS AND ITS EVIDENCE

Before formulating and arguing the central affirmation, that motivates this exploration, we need one more trivial technical unnumbered claim, which is included only for the sake of completeness.

As usual, $V(RG)$ designates the normed unit group in RG.

Lemma. *If $G \cong H$, then $V(RG)/G \cong V(RH)/H$ and $S(RG)/G_p \cong S(RH)/H_p$.*

Proof. We shall verify only the first relationship because the second one is similar.

By hypothesis there is an isomorphism $\varphi: G \to H$. Next, we define a map $\Phi: V(RG) \to V(RH)$, which linearly extended φ, in the following manner: $\Phi(\sum_i r_i g_i) = \sum_i r_i \varphi(g_i)$. It is straightforward that Φ is an isomorphism such that $\Phi_G = \varphi$. Consequently, it is directly seen that Φ induces the isomorphism $\psi: V(RG)/G \to V(RH)/H$ defined like this $\psi[(\sum_i r_i g_i)G] = [\Phi(\sum_i r_i g_i)]H$, which substantiates the claim. The proof is completed.

We are now able to proceed by proving the following crucial tool.

Proposition. $RG \cong RH \Rightarrow R(S(RG)/G_p) \cong R(S(RH)/H_p)$.

Proof. Certainly, $RG = RH'$ for some $H' \leq V(RG)$ so that $H' \cong H$. According to the Lemma, we derive $S(RH')/H'_p \cong S(RH)/H_p$.

Besides, it is apparent that $S(RG) = S(RH')$. Denote this group by A. Therefore (see e.g. [1], [7], [8] and [10]), $I(RG; G_p) = I(RH'; H'_p)$, hence $I(RA; G_p) = RA.I(RG; G_p) = RA.I(RH'; H'_p) = I(RA; H'_p)$. Finally, $R(A/G_p) \cong RA/I(RA; G_p) = RA/I(RA; H'_p) \cong R(A/H'_p)$. So, by what we have concluded above, $R(S(RG)/G_p) \cong R(S(RH)/H_p)$ and thus we are done. The proof is complete.

Now, we are in a position to prepare the following.

Theorem. *Suppose G is an abelian group and suppose $RG \cong RH$ are isomorphic R-algebras over an arbitrary perfect field R of characteristic $p>0$ and for some arbitrary group H. If G_p belongs to at most one of the classes of*

(a) *totally projective (simply presented) groups;*

(b) *$p^{\omega+n}$-projective groups ($n \in IN$);*

(c) *summable groups with countable lengths;*

(d) *σ-summable groups and their direct sums;*

(e) *Σ-groups,*

then both $S(RG)/G_p$ and $S(RH)/H_p$ belong to these group sorts, additionally provided that G is a p-group when (a) and the second part half of (d) hold. Moreover, in the cases (a) and (b), it is fulfilled $S(RG)/G_p) \cong S(RH)/H_p$, extra assuming that R is finite when (b) holds.

Proof. Referring to the preceding Proposition, we infer that $R(S(RG)/G_p) \cong R(S(RH)/H_p)$. Now, we wish to employ the corresponding affirmations from [18], [15], [14, 16], [1, 11] and [3] to get the claim. The proof is finished.

Corollary. *Let G be an abelian group and let $RG \cong RH$ be R-isomorphic for a group H over a perfect field with nonzero characteristic p. If $S(RG)/G_p$ lies in one of the following classes of*

(a') totally projective (simply presented) groups;

(b') $p^{\omega+n}$-projective groups ($n \in IN$);

(c') summable groups with countable lengths;

(d') σ-summable groups and their direct sums;

(e') Σ-groups,

then so does $S(RH)/H_p$. Besides, in the cases (a') and (b'), it is fulfilled that $S(RG)/G_p) \cong S(RH)/H_p$, provided additionally that R is finite when (b') holds.

Proof. It follows by the same token as demonstrated above.

After this, we are ready to attack the applications.

In the proof of statements from [4], [5], [9] and [12] we have to check that the *p*-group *H* is a direct factor of *V(RH)* presuming that so is *G* for *V(RG)* with the totally projective complementary factor *V(RG)/G*. Certainly, with the aid of the foregoing stated machinery plus certain folklore arguments, it is not difficult to do this. Nevertheless, for a convenience of the readers, we shall concern this matter detailed. The formulations are the same as in the cited below bibliography.

Theorem 3.1. ([4]). *Let G be a direct sum of torsion-complete abelian p-groups with cardinality of each factor not exceeding $\aleph_1$. Then $IK_pH \cong IK_pG$ as IK_p-algebras for some group H and over the p-element field IK_p if and only if $H \cong G$.*

Proof. Indeed, because $V(IK_pG)/G$ is totally projective (see e.g. [4]) and $IK_pG \cong IK_pH$, a routine appeal to the preceding established point (a) means that $V(IK_pH)/H$ is totally projective. But we have previously remarked that *H* is balanced in $V(IK_pH)$, hence this insures the wanted direct factor property and the proof goes similarly to [4]. The proof is over.

Major Theorem (Isomorphism) ([12]). *Suppose G is a direct sum of groups so that their p-components are of countable length direct sums of countable groups and $FH \cong FG$ as F-algebras for any group H and over a perfect field of $char(F) = p \neq 0$. Then $H_p \cong G_p$. Besides, there is a direct sum of p-countable groups T with the property $H \times T \cong G \times T$ provided the torsion subgroup G_t is p-torsion.*

Proof. The first part half follows as in [12].

Next, we concentrate on the second one. And so, as in [12], $S(FG)/G_p$ is simply presented and *G* is a direct factor of *V(FG)* such that $V(FG)/G \cong S(FG)/G_p$. Furthermore, point (a') is a guarantor that $S(FH)/H_p \cong S(FG)/G_p$ is simply presented. Consequently, H_p is a direct factor of *S(FH)*, whence *H* is so for *V(FH)* since it is trivial that H_t is a *p*-group and thereby $V(FH) = HS(FH)$. Finally, $G \times S(FG)/G_p \cong H \times S(FH)/H_p \cong H \times S(FG)/G_p$, and by the putting $T = S(FG)/G_p$ everything is proved.

We close the investigation with

Analysis. The evidences of the above two assertions unambiguously demonstrate that another method for confirmation was deduced. While in [4] we must showed that *H* is special direct sum of *p*-primary groups such as is *G*, that is not so elementary, here we have used the more easy argumentation on the isomorphism structure of $S(RH)/H_p$ which is in a natural connection with that of $S(RG)/G_p$ by preserving the structural properties.

The same applies for [12], although there we not intended to yield that H is a direct sum of groups equipped with the special properties such as is G. Incidentally, there we have exploited different isomorphism relation, namely $G_p \cong H_p$ whenever $FG \cong FH$.

Corrigendum. In [15] there are four printer misprints. In fact, the sign "=" on p. 259, line 11(-); p. 264, line 6(-); p. 267, line 4(-), the third equality; p. 268, line 8(+), the second equality, should be read as "$\neq$". Moreover, on p. 266 there is an omission, namely $G_k \subseteq M_k \subseteq M_{k+1}$ written on line 15(+) must be replaced by $G_k \cap G^{p\omega} \subseteq M_k \subseteq M_{k+1}$.

REFERENCES

1. DANCHEV, P., *Commutative group algebras of σ-summable abelian groups*, Proc. Amer. Math. Soc., **125**, 9, pp. 2559-2564, 1997.
2. DANCHEV, P., *Isomorphism of commutative group algebras of mixed splitting groups*, Compt. rend. Acad. bulg. Sci., **51**, 1-2, pp. 13-16, 1998.
3. DANCHEV, P., *Commutative group algebras of abelian Σ-groups*, Math. J. Okayama Univ., **40**, 2, pp. 77-90, 1998.
4. DANCHEV, P., *Isomorphism of modular group algebras of direct sums of torsion-complete abelian p-groups*, Rend. Sem. Mat. Univ. Padova, **101**, pp. 51-58, 1999.
5. DANCHEV, P., *Modular group algebras of coproducts of countable abelian groups*, Hokkaido Math. J., **29**, 2, pp. 255-262, 2000.
6. DANCHEV, P., *Invariances in commutative and noncommutative group algebras*, Compt. rend. Acad. bulg. Sci., **54**, 4, pp. 5-8, 2001.
7. DANCHEV, P., *Numerical invariants for commutative group algebras*, Ricerche Mat., **50**, 2, pp. 323-336, 2001.
8. DANCHEV, P., *Invariants for group algebras of abelian groups*, Rend. Circolo Mat. Palermo, **51**, 3, pp. 391-402, 2002.
9. DANCHEV, P., *Commutative group algebras of cardinality* $\aleph_1$, Southeast Asian Bull. Math., **25**, 4, pp. 589-598, 2002.
10. DANCHEV, P., *Numerical invariants for modular group algebras of abelian groups*, Ricerche Mat., **52**, 2, pp. 197-202, 2003.
11. DANCHEV, P., *Commutative group algebras of direct sums of σ-summable abelian p-groups*, Math. J. Okayama Univ., **45**, 2, pp. 1-15, 2003.
12. DANCHEV, P., *Commutative group algebras of direct sums of countable abelian groups*, Kyungpook Math. J., **44**, 1, pp. 21-29, 2004.
13. DANCHEV, P., *Ulm-Kaplansky invariants for* $S(RG)/G_p$, Bull. Inst. Math. Acad. Sinica, **32**, 2, pp. 133-144, 2004.
14. DANCHEV, P., *On a question of W. L. May concerning the isomorphism of modular group algebras*, Comm. Algebra, **29**, 5, pp. 1953-1958, 2001.
15. DANCHEV, P., *Commutative group algebras of* $p^{\omega+n}$*-projective abelian groups*, Acta Math. Vietnamica, **29**, 3, pp. 259-270, 2004.
16. DANCHEV, P., *Commutative group algebras of summable p-groups*, Comm. Algebra, **34**, 2006.
17. MAY, W., *Modular group algebras of totally projective p-primary groups*, Proc. Amer. Math. Soc., **76**, 1, pp. 31-34, 1979.
18. MAY, W., *Modular group algebras of simply presented abelian groups*, Proc. Amer. Math. Soc., **104**, 2, pp. 403-409, 1988.

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