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ON BOOLEAN PRODUCT OF COMPLETELY POSITIVE MAPS

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We prove directly that the Boolean product of contractive, completely positive maps is completely ositive on the non-unital full free product C*-algebra and list certain consequences of this fact.

Key words: universal free product (C*-)algebra, Boolean product, complete positivity, complete contractivity, positive definite function, spectral set.

1. INTRODUCTION

Corresponding to the Boolean independence originated in W. von Waldenfels' work on the pressure broadening of spectral lines (see, e.g., [16]), the Boolean product of linear functionals on algebras is defined on the associated universal free product algebra without unit, and on involutive algebras it preserves the positivity (see, e.g., [12]).

This product and the involved independence receive increasing attention in the noncommutative probability, after R. Speicher [13] answered to M. Schürmann's conjecture [12] on the universal products of algebraic probability spaces.

Besides the tensor product going to Boson or Fermion independence used by C. D. Cushen and R. L. Hudson [7] [8], and the free product corresponding to the concept of free independence (freeness) due to D. Voiculescu (see, e.g., [15]), Speicher pointed out in the non-unital case there exists only one supplementary product, namely the Boolean product, which does not depend on the order of its factors, is associative and fulfills a universal rule for mixed moments (see [13]).

Then, A. Ben Ghorbal and Schürmann [2] completed Speicher's answer by the classification of all universal products in the sense in [12].

In this Note, we consider the Boolean product for linear maps between algebras and show, by a direct proof, it preserves the complete positivity in C*-algebraic setting. Consequences of this fact are listed by analogy with some of F. Boca's results concerning the free product [3] [4].

2.BOOLEAN PRODUCT OF LINEAR MAPS

Letting A be a (complex) *-algebra (i.e., a complex algebra with a conjugate linear involution *, which is an anti-isomorphism), a positive element in A is a finite sum $\sum a_i^* a_i$, with $a_i \in A$. The set A_+ of positive elements in A determines a preorder structure on the real linear subspace of self-adjoints elements in A.

Let S be a subset of A, B be another *-algebra and $Q : S \longrightarrow B$ be a map. We say Q is positive if $Q(S \cap A_+) \subset B_+$.

If $n \in \mathbf{N}$, the set of positive integers, let $M_n(A)$ be the *-algebra of $n \times n$ matrices $[a_{ij}]$ with

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entries from A, $M_n(S) \subset M_n(A)$ be the set of $n \times n$ matrices over S, and $Q_n : M_n(S) \longrightarrow M_n(B)$ be the map given by $Q_n([s_{ij}]) = [Q(s_{ij})]$, for $[s_{ij}] \in M_n(S)$. Then Q is called *n*-positive, if the map Q_n induced by Q is positive. The map Q is completely positive, if it is *n*-positive, for all $n \in \mathbf{N}$.

For examples of *n*-positive maps between C*-algebras that fail to be $(n+1)$-positive, if $n \geq 1$, see, e.g. , [6].

Whenever A and B, as before, are endowed with seminorms, $S \subset A$ is a linear subspace, and Q is a linear map, Q is called completely contractive, if all Q_n are contractive; endowing $M_n(S)$ with the seminorm that it inherits from $M_n(A)$.

In particular, when A is a C*-algebra, A_+ determines an order structure on the subspace of self-adjoints elements in A, and we let $M_n(S)$ have the norm and order that it inherits from the C*-algebra $M_n(A)$.

The universal free product (*-, C*-)algebra is the direct sum in the category of (complex) (*-, C*-) algebras, non necessary unital [1][2][3][4][11][12][15].

We denote, as usually, by $*_1 A_i$, respectively, $*_0 A_i$, the unital, respectively, non-unital, universal (or full) free product C*-algebra corresponding to an adequate family $(A_i)_{i \in I}$ of C*-algebras.

As linear space, a realization of the universal free product associated to a family of (*-, C*-)algebras $(A_i)_{i \in I}$ is

$$A = \oplus_{n \geq 1} \oplus_{i_1 \neq \cdots \neq i_n} A_{i_1} \otimes \ldots \otimes A_{i_n} .$$

By natural operations, A is organized as a (*-)algebra.

In particular, if A_i are C*-algebras, A satisfies the Combes axiom, i.e. for every $a \in A$, there exists a scalar $\lambda(a) > 0$ with $x^* a^* a x \leq \lambda(a) x^* x$, for all $x \in A$.

After separation and completion of the corresponding universal free product *-algebra A in its enveloping C*-seminorm $\|a\| = \sup\{\|\pi(a)\|; \pi \ * - \text{representation of } A \text{ on a Hilbert space}\}$, one can realize the universal (or full) free product $*_0 A_i$ in the category of C*-algebras.

Let B and A_i be (complex) algebras, and $Q_i : A_i \longrightarrow B$ be linear maps; $i \in I$.

We consider the Boolean product $Q = \bullet Q_i$ as the unique linear map defined on the universal free product A of the algebras A_i, $i \in I$, such that

$$Q(a_1 \ldots a_n) = Q_{i_1}(a_1) \ldots Q_{i_n}(a_n) ,$$

for all $n \geq 1$, with $i_1 \neq \ldots \neq i_n$, and $a_k \in A_{i_k}$, if $k = 1,\ldots,n$; relatively to the natural embeddings of A_i into A arising from the free product construction.

When $B = \mathbf{C}$, the complex field, we obtain the Boolean product of functionals in [2][12].

It is easy to check that the Boolean product of linear maps is associative.

It was remarked in [3] that the classical Stinespring dilation theorem (see, e.g., [10]) is still true for unital completely positive maps on unital *-algebras verifying the Combes axiom.

Theorem 1. *Let A_i and B be C*-algebras, and $Q_i : A_i \longrightarrow B$ be contractive, completely positive maps; $i \in I$.*

*Then the Boolean product $Q = \bullet Q_i$ defined on the *-algebraic free product of the algebras A_i, $i \in I$, is contractive, with respect to the enveloping C*-seminorm, and completely positive.*

Therefore, Q extends to a unique map $Q : *_0 A_i \longrightarrow B$ which is contractive and completely positive.

Sketch of the proof. Adapting Boca's ideas from [3], the theorem is a consequence of the next lemma through a Stinespring construction.

Lemma 2. *Suppose that A_i and B are C*-algebras, and that $Q_i : A_i \longrightarrow B$ are contractive, completely positive maps; $i\in I$. Denote by A the *-algebraic free product of $(A_i)_{i\in I}$ and by $Q : A \longrightarrow B$ the Boolean product of $(Q_i)_{i\in I}$.*

If we let A_1 and B_1 be the unital extensions of A and B by **C**, *the unital map $\mathrm{Q}_1 : \mathrm{A}_1 \longrightarrow \mathrm{B}_1$ defined by $Q_1(a \oplus \lambda 1) = Q(a) \oplus \lambda 1$ ($a\in A$, $\lambda \in$ **C**) is completely positive.*

Sketch of the proof of this lemma. Assume that $B \subseteq B_1 \subseteq L(H)$, with a Hilbert space *H*.

Denote by $W = \{1\} \cup \{a_1 ... a_n;\ n \geq 1,\ a_k \in A_{i_k},\ i_1 \neq ... \neq i_n\}$ the set of reduced words in A_1.

For $w = a_1 ... a_n \in W$, $w \neq 1$, call n the length of w and denote $\tilde{w} := \{1, a_1, a_1 a_2, ..., a_1 a_2 \cdots a_n\}$.

The length of the empty word 1 is zero and $\tilde{1} = \{1\}$.

Calling a subset of W complete if it contains 1 and it includes $\tilde{w}$, whenever it contains a word *w*, it is enough to check that $S_X^f := \sum_{x,y\in X} < Q_1(x^* y) f(y), f(x) > \geq 0$, for all complete finite sets $X \subset W$ and all maps $f : X \to H$.

1^0 If such a finite set X has $k \geq 3$ words, choose a word $a_1 \cdots a_m$ of maximum length in *X*, with $a_k \in A_{i_k}$, for *k=1,…,m*; and $i_1 \neq ... \neq i_m$.

Then the subsets $X_2 = X \cap A_{i_1} \otimes \cdots \otimes A_{i_m}$ and $X_1 = X \setminus X_2$ form a partition of *X*, in which X_1 is complete, having *n*, less than *k*, words.

When $S_{X_1}^g \geq 0$, for all maps $g : X_1 \to H$, remark that there exists $V_x \in L(H, H^{\oplus n})$ such that $Q_1(x^* y) = V_x{}^* V_y$, if $x, y \in X_1$*;* and, $[V_x^* V_y]_{x,v\in X_1} \geq [V_x^* V_1 V_1^* V_y]_{x,y\in X_1}$ in $M_n(L(H)) = L(H^{\oplus n})$.

1^0.1 When $X_2 = X \cap A_i$, $i \in I$, note that $Q_1(x^* y) = Q_1(x)^* Q_i(y)$, for $x \in X_1$, and $y \in X_2$.

In this way, it is enough to observe that the $(l+1)$-positivity and contractivity of Q_i implies $[Q_i(x^* y)]_{x,y\in X_2} \geq [Q_i(x)^* Q_i(y)]_{x,y\in X_2}$ in $M_l(L(H)) = L(H^{\oplus l})$*;* denoting by *l* the number of words in X_2.

Therefore, we may deduce

$$S_X^f = S_{X_1}^f + \sum \ + S_{X_2}^f \geq \left\| \sum_{x\in X_1} Q_1(x) f(x) \right\|^2 + 2\,\mathrm{Re}\left\langle \sum_{x\in X_1} Q_1(x) f(x), \sum_{x\in X_2} Q_i(x) f(x) \right\rangle + \left\| \sum_{x\in X_2} Q_i(x) f(x) \right\|^2 =$$

$$= \left\| \sum_{x\in X_1} Q_1(x) f(x) + \sum_{x\in X_2} Q_i(x) f(x) \right\|^2,$$

whenever $S_{X_1}^g \geq 0$, for all $g : X_1 \longrightarrow H$.

1^0.2 Otherwise, note that every word $x \in X_2$ may be written as $x_o a$ with $x_o \in X_1$, $x_o \neq 1$, and $a \in A_{i_m}$.

Therefore $Q_1(x^* y) = Q_{i_m}(a)^* Q(x_o^* y)$, for $x = x_o a \in X_2$, and $y \in X_1$*;* similarly, $Q_1(x^* y) = Q_{i_m}(a')^* Q(x_o'^* x_o) Q_{i_m}(a)$, for $x = x_o' a' \in X_2$, and $y = x_o a \in X_2$.

Consequently, we may deduce

$$S_X^f = S_{X_1}^f + \sum \ + S_{X_2}^f = \left\| \sum_{x\in X_1} V_x f(x) \right\|^2 + 2Re\left\langle \sum_{x\in X_1} V_x f(x), \sum_{y=x_o a\in X_2} V_{x_o} Q_{i_m}(a) f(y) \right\rangle + \left\| \sum_{x=x_o a\in X_2} V_{x_o} Q_{i_m}(a) f(x) \right\|^2 =$$

$$= \left\| \sum_{x\in X_1} V_x f(x) + \sum_{x=x_o a\in X_2} V_{x_o} Q_{i_m}(a) f(x) \right\|^2$$

whenever $S^{g}_{X_1} \geq 0$, for all $g : X_1 \to H$.

2^0 If $X = \{1, a\}$, with $a \in A_i$, $i \in I$, it is enough to observe that Schwarz type inequality for 2-positive maps (see, e.g., [10]) allows the positivity in $M_2(L(H)) = L(H^{\oplus 2})$ of $\begin{bmatrix} 1 & Q_i(a) \\ Q_i(a)^* & Q_i(a^*a) \end{bmatrix}$.

The following is a Boolean analogue of Corollary 4 in [4], being another noncommutative version of Th. 10.8 in [10] (or Prop. 4.23 in [14]).

Corollary 3. *Let Q_i be contractive, completely positive maps between the C*-algebras A_i and B_i, $i \in I$.*

There is a common extension $Q = \bullet Q_i : *_0 A_i \longrightarrow *_0 B_i$ which is contractive and completely positive, satisfying

$$Q(a_1 \ldots a_n) = Q_{i_1}(a_1) \ldots Q_{i_n}(a_n),$$

for all $n \geq 1$, with $i_1 \neq \ldots \neq i_n$, and $a_k \in A_{i_k}$, if $k = 1, \ldots, n$; with respect to the natural embeddings of A_i into $*_0 A_i$ and, respectively, of B_i into $*_0 B_i$, arising from the free product constructions.

The next corollary describes properties of the Boolean product concerning classes of completely positive maps; for embeddings, the shortest way is via [11, Th. 4.2](see also [1, Prop. 2.2]).

Corollary 4. *Let A_i and B_i be C*-algebras, $i \in I$.*

*If π_i is a *-homomorphism between A_i and B_i, then $\pi = \bullet \pi_i$ is the unique *-homomorphism $\pi = *_0 \pi_i$ between the C*-algebras $*_0 A_i$ and $*_0 B_i$, such that $\pi|A_i = \pi_i$, for each $i \in I$, with respect to the mentioned embeddings.*

If π_i are embeddings, then $\pi = \bullet \pi_i$ is an embedding.

*If π_i is a *-automorphism of A_i, then $\pi = \bullet \pi_i$ is a *-automorphism of the C*-algebra $*_0 A_i$.*

If B_i is a C-subalgebra of A_i, and E_i is a conditional expectation of A_i onto B_i, $i \in I$, then the Boolean product $E = \bullet\, E_i$ is a conditional expectation of the C*-algebra $*_0 A_i$ onto its C*-subalgebra $*_0 B_i$.*

In the sequel, *L(H)* denotes, as above, the bounded linear operators on a Hilbert space *H*.
If A is a unital C*-algebra, an operator system X in A is a self-adjoint linear subspace $X \subset A$ containing the unit of A (see, e.g., [10]).

***Corollary** 5. Let A_i be unital C*-algebras, $X_i \subset A_i$ be operator systems, and $\Phi_i : X_i \to L(H)$ be unital completely positive maps; $i \in I$.*

Then, there exists a contractive, completely positive map on the C-algebra $*_0 A_i$ extending each Φ_i.*

Proof. By Arveson's extension theorem (see, e.g., [10]), each Φ_i extends to a unital completely positive map $Q_i : A_i \to L(H)$, and, by Theorem 1, the Boolean product $Q = \bullet Q_i$ is contractive and completely positive on $*_0 A_i$.

A unital version of the following fact was pointed out in [3][4] . In our case, the extension does not require the choice of auxiliary states on the involved algebras.

Corollary 6. *Let A_i be unital C*-algebras, $S_i \subseteq A_i$ be unital linear subspaces and $L_i : S_i \longrightarrow L(H)$ be unital completely contractive maps; $i \in I$.*

Then there is a completely contractive map on the C-algebra $*_0 A_i$ extending every L_i.*

Proof. Each L_i extends, via Prop. 3.4 in [10], to a unital completely positive map $\Phi_i : X_i \to L(H)$, where X_i is the operator system $S_i + S_i^*$. By the previous corollary, each Φ_i extends to a contractive and completely positive map on $*_0 \mathrm{A}_i$.

Let G be a locally compact group, and $C^*(G)$ be the associated (full) group C*-algebra. For $g \in G$, denote by δ_g the elementary function taking the value 1 on *g*, and null, in rest.

There is a one-to-one correspondence between the weakly continuous positive-definite operator-valued functions on G, and the completely positive maps on $C^*(G)$ (see, e.g., [10, 4.11]).

Boca [4] stated the unital free analogue of the following assertion. In this way, he recovered the free product of positive definite functions constructed by M. Bożejko [5] as an extension of Haagerup's lemma concerning the length function on the free group; and, in [3, Prop. 4.1], he extended the noncommutative von Neumann inequality due to Bożejko [5].

Corollary 7. *Let* $\mathrm{G_i}$ *be locally compact groups,* $\varphi_i : \mathrm{G_i} \longrightarrow \mathrm{L(H)}$ *be unital, weakly continuous positive definite functions, and* $Q_i : C^*(G_i) \longrightarrow L(H)$ *be completely positive maps such that* $Q_i(\delta_g) = \varphi_i(g)$, *if* $g \in G_i$; $i \in I$.

Then the Boolean product $Q = \bullet Q_i : *_0 C^*(G_i) \longrightarrow L(H)$ *is still completely positive.*

Let Ω be a compact set in the complex plane and let $R(\Omega)$ be the algebra of rational functions which are analytic on Ω, embedding $R(\Omega)$ as a subalgebra of $C(\partial\Omega)$, the continuous functions on the boundary of Ω, by the maximum modulus theorem: this endows $R(\Omega)$ with the norm $\|f\| = \sup_{z\in\Omega} |f(z)| = \sup_{z\in\partial\Omega} |f(z)|$.

If T is a bounded linear operator on a Hilbert space H, and the spectrum of T is contained in Ω, there exists a unital homomorphism $\rho_T : R(\Omega) \longrightarrow L(H)$ given by an elementary functional calculus $\rho_T(f) = f(T)$, for $f \in R(\Omega)$.

When ρ_T is a (completely) contractive map, Ω is called a (completely) spectral set for *T*. The spectral sets were introduced by J. von Neumann whose famous inequality means that an operator is a contraction if and only if the closed unit disk is a spectral set for it; moreover, this is a complete spectral set, for all contraction, by the celebrated unitary dilation theorem due to B. Sz.-Nagy (see, e.g., [10]).

If Ω is a spectral set for *T*, then there is a well-defined positive map $\tilde{\rho}_T$ on the operator system $R(\Omega) + R(\Omega)^*$ in $C(\partial\Omega)$, given by $\tilde{\rho}_T(f + g^*) = \rho_T(f) + \rho_T(g)^*$, for $f, g \in R(\Omega)$ (see, e.g., [10, Prop.2.12]).

Let Ω_i be spectral sets for $T_i \in L(H)$; $i \in I$.

Denote $S := \{ f \in *_0 C(\partial\Omega_i);\ f = \sum_{i_1 \neq \ldots \neq i_n} f_1 \ldots f_n,$ where $n \geq 1$, and $f_k \in R(\Omega_{i_k}) + R(\Omega_{i_k})^* \}$, in parallel with [3, section 4].

For $f = \sum_{i_1 \neq \ldots \neq i_n} f_1 \ldots f_n \in S$, denote also $f((T_i)) = \sum_{i_1 \neq \ldots \neq i_n} \tilde{\rho}_{T_{i_1}}(f_1) \ldots \tilde{\rho}_{T_{i_n}}(f_n) \in L(H)$.

Moreover, for each $m \in \mathbf{N}$, if $f = [f_{kl}] \in M_m(S)$, denote $f((T_i)) := [f_{kl}((T_i))] \in M_m(L(H))$.

Actually, the Boolean product yields a noncommutative von Neumann type inequality of the following matrix form.

Corollary 8. *Let* Ω_i *be complete spectral sets for* $T_i \in L(H)$, $i \in I$.

Then, for every $m \in \mathbf{N}$, *and* $f \in M_m(S)$:

$$\| f((T_i)_i) \| \leq \| f \|_{M_m(*_0 C(\partial\Omega_i))}$$

Proof. The complete contractivity of the unital homomorphisms ρ_{T_i} implies the complete positivity of the maps $\widetilde{\rho}_{T_i}$ as in Corollary 6. As in Corollary 5, every $\widetilde{\rho}_{T_i}$ extends to a unital completely positive map $Q_i : C(\partial\Omega_i) \longrightarrow L(H)$; and, the Boolean product $Q = \bullet Q_i$ is completely contractive.

For all $f = \sum_{i_1 \neq \dots \neq i_n} f_1 \dots f_n \in S$, it results, as in [3]:

$$f((T_i)) = \sum_{i_1 \neq \dots \neq i_n} \tilde{\rho}_{T_{i_1}}(f_1) \dots \tilde{\rho}_{T_{i_n}}(f_n) = \sum_{i_1 \neq \dots \neq i_n} Q_{i_1}(f_1) \dots Q_{i_n}(f_n) = Q(f).$$

Therefore, for every $f = [f_{kl}] \in M_m(S)$, one obtains $f((T_i)) = [f_{kl}((T_i))] = [Q(f_{kl})] = Q_m(f)$, with $Q_m \equiv Q \otimes I_m : M_m(*_0 C(\partial\Omega_i)) \longrightarrow M_m(L(H))$, the maps induced by Q, and

$$\| f((T_i)) \| = \| Q_m(f) \| \le \| f \|_{M_m(*_0 C(\partial\Omega_i))}.$$

Remark 9. In fact, it is easy to see, the *complete*-contractivity of Boca's unital free product implies similarly the following matrix von Neumann type inequality which is implicit in [3, Prop. 4.1] (see also [9]).

Let Ω_i be complete spectral sets for $T_i \in L(H)$, $i \in I$, specifying a probability measure μ_i on every $\partial\Omega_i$. If $m \in \mathbf{N}$, and $f \in M_m(S(*\Omega_i))$, then

$$\| f((T_i)_i) \| \le \| f \|_{M_m(*_1 C(\partial\Omega_i))},$$

where $S(*\Omega_i) := \{ f = \alpha \cdot 1 + \sum_{i_1 \neq \dots \neq i_m} f_1 \dots f_m \in *_1 C(\partial\Omega_i);\ f_k \in C(\partial\Omega_{i_k})^0,\ f_k \in R(\Omega_{i_k}) + R(\Omega_{i_k})^*,\ \alpha \in \mathbf{C} \}$;

denoting by $C(\partial\Omega_i)^0$ the kernel of the state on $C(\partial\Omega_i)$ corresponding to μ_i; for $f = \alpha \cdot 1 + \sum_{i_1 \neq \dots \neq i_m} f_1 \dots f_m \in S(*\ \Omega_i)$, setting $f((T_i)) = \alpha \cdot I_H + \sum_{i_1 \neq \dots \neq i_m} \widetilde{\rho}_{T_{i_1}}(f_1) \dots \widetilde{\rho}_{T_{i_m}}(f_m)$; with an extension of Boca's notation to square matrices over $S(*\Omega_i)$.

Whenever T_i are n contractions, Ω_i are the closed unit disk for each $i = 1, \dots, n$, letting on $\partial\Omega_i$ the Haar measure, and f is a $m \times m$ matrix whose entries are free polynomials in at most n noncommutative indeterminantes, the above inequality becomes

$$\| f((T_i)_i) \| \le \| f \|_{M_m(C^*(F_n))},$$

denoting by F_n the free group on n generators, $n \in \mathbf{N} \cup \{\infty\}$; for $m = 1$ and $n \in \mathbf{N}$, it reduces to Bozejko's von Neumann inequality in [5].

Other facts including the Stinespring dilation for the map in Theorem 1, and limit theorems for extended Boolean random variables will be presented in a forthcoming paper. These limit theorems confirm the connection revealed in [13] between the Boolean product of noncommutative probability spaces and the interval partitions.

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