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REVISITING SEISMIC ISOLATION FROM A MODAL ENERGY PERSPECTIVE

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The purpose of this paper is to determine the structural dynamic properties of seismically isolated buildings, idealized as two-degree-of-freedom systems, using an analytical formulation based on the modal kinetic and strain energy ratios of such systems. As a result of these energy considerations the expressions derived are compact, and allow for a renewed visualization of the salient behavior of isolated structures. A brief numerical comparison with similar previous equations by others demonstrates the equivalency and validity of the expressions shown in the present paper.

Key words: seismic isolation, structural design, building codes.

1. INTRODUCTION

Early in the development of seismic isolation, the dynamic behavior of buildings thus equipped was the subject of significant study. Miranda [1] at the Centre National de la Recherche Scientifique (C.N.R.S.) in France, found that for elastic structural systems the presence of isolators significantly modified the properties of the fundamental mode, and introduced a new second mode. He also found that the influence of the isolators on the higher modes was less significant, and that such influence decreased with the order of the modes. Similarly, he found that due to the negligible participation factors of the higher modes, the response was mostly associated with the first mode. At U.C. Berkeley, Kelly [2] demonstrated that the fundamental behavior of seismically isolated elastic structures could be captured using two-degree-of-freedom mechanical systems idealizations. Recently, Miranda [3] developed a modal energy based analytical model for two-degree-of-freedom systems, and applied it to the study of tuned mass dampers for seismic response reduction. In this paper we use the latter model as an analytical platform to re-calculate, from the energy perspective, the structural dynamic characteristics of seismically isolated structures; mode shapes, frequencies, participation factors, and damping, as once furnished by Kelly [2]. As a collateral result, and due to its energy formulation, we shall see that the model provides physical insight on the behavior of these structures. In addition, we shall see that the simplicity of the analytical model facilitates a compact derivation of the pertinent properties. Finally, a single numerical application shows the equivalency and validity of the results from this paper with the results obtained by Kelly [2] through traditional means.

2. CALCULATION OF STRUCTURAL DYNAMIC PROPERTIES

We consider the two-degree-of-freedom mechanical system shown on Figure 1, wherein the upper portion has a mass M_U, a viscous linear damper with constant C_U, and a spring with linear horizontal stiffness K_U. These parameters represent the effective properties of the superstructure of a seismically isolated building. A mass M_L, a viscous linear damping constant C_L, and a linear horizontal stiffness K_L, similarly characterize the lower portion. The latter parameters represent the properties of the isolated base of the building. The following variables, all known, are defined:

$$\omega_U^2 = \frac{K_U}{M_U} \tag{2.1}$$

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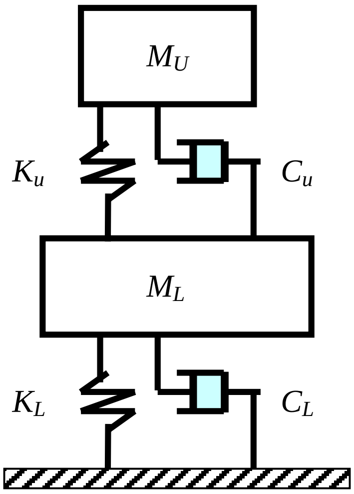

Fig.1. Two-Degree-of-Freedom Mechanical System

$$\omega_L^2 = \frac{K_L}{M_L} \quad (2.2)$$

$$\Omega = \frac{\omega_U}{\omega_L} \quad (2.3)$$

$$\mu = \frac{M_U}{M_L} \quad (2.4)$$

$$\omega_I^2 = \frac{K_L}{M_U + M_L} \quad (2.5)$$

$$\delta = \frac{\omega_I}{\omega_U} \quad (2.6)$$

Equation (2.1) provides the circular frequency of the superstructure when considered in its fixed base condition. Equation (2.2) provides the circular frequency of the isolated base when considered independently. Equation (2.3) represents a ratio between the circular frequencies of the fixed base superstructure and the isolated base. Equation (2.4) is the ratio of the mass of the superstructure with respect to the mass of the isolated base. Equation (2.5) provides the circular frequency for a single-degree-of-freedom mechanical system constituted by an infinitely rigid superstructure plus base, placed on the isolation system. For simplicity, from herein on we refer to this particular mechanical system as the "single-degree-of-freedom isolated superstructure". Equation (2.6) represents a ratio between the frequencies of the isolated single-degree-of-freedom building, and of the fixed base superstructure. It is noted that in practice the ratio expressed by equation (6) is usually of the order of 0.1.

Miranda [3], derived the dynamic properties of two-degree-of-freedom systems based on their modal kinetic and strain energies. A succinct presentation of this work is given in appendix. Per the theoretical development therein, we define the ratios α_j and β_j of the modal kinetic and strain energies assumed by the superstructure with respect to the corresponding energies assumed by the isolated base, while the mechanical system vibrates freely in its X_j mode. The two α_j and the two β_j parameters are interrelated in such a way that the determination of any of them determines the other three. Subsequently, the mode shapes, frequencies, participation factors, and an approximation to modal damping are found as functions of these parameters. We proceed as follows:

In the case of properly seismically isolated buildings, it is well known that the fundamental mode exhibits little drift. Hence, following the theory in appendix if the modal component at the isolation level is normalized to one, we can write that the component corresponding to the superstructure is approximately:

$$\left(\frac{\alpha_1}{\mu}\right)^{1/2} = 1 + \varepsilon \quad (2.7)$$

where ε is a small quantity to be determined. Due to the orthogonality of the modes with respect to the mass matrix, we have:

$$\alpha_2 = \frac{1}{\alpha_1} \quad (2.8)$$

And due to the orthogonality of the modes with respect to the stiffness matrix we have in addition:

$$\Omega^2 = \frac{1}{1 + \sqrt{\mu\alpha_1} - \sqrt{\mu\alpha_2} - \mu} \quad (2.9)$$

Replacing equation (2.7) and (2.8) into (2.9), and ignoring higher order values of ε results in:

$$\Omega^2 = \frac{1+\varepsilon}{\varepsilon(1+\mu)} \tag{2.10}$$

and hence:

$$\varepsilon = \frac{1}{\Omega^2(1+\mu)-1} \tag{2.11}$$

Using equation (2.5) and (2.6), equation (2.11) becomes finally:

$$\varepsilon = \frac{\delta^2}{1-\delta^2} \tag{2.12}$$

We verify from equation (2.12) that ε is of the order of 0.01. We can then determine the modal kinetic energy ratios as:

$$\alpha_1 = \frac{\mu}{\left(1-\delta^2\right)^2} \tag{2.13}$$

and

$$\alpha_2 = \frac{\left(1-\delta^2\right)^2}{\mu} \tag{2.14}$$

Equation (2.13) reveals that in the fundamental mode the velocities of the superstructure and of the isolated base are approximately the same, that is:

$$\dot{X}_{U1} \cong \dot{X}_{L1} \tag{2.15}$$

where, in accordance with the theory shown in appendix, the subscripts U and L indicate upper and lower modal components. This characteristic evidences the quasi-rigid fundamental mode shape typical of properly isolated buildings. This is, of course, the shape that we have assumed at our departing state. Equation (2.14) reveals that in this mode the velocities of the superstructure and of the isolated base are approximately related as:

$$\dot{X}_{U2} \cong -\frac{1}{\mu}\dot{X}_{L2} \tag{2.16}$$

This characteristic indicates an also quasi-rigid second mode shape, with the superstructure moving slower than the isolated base. We finally write the modes shapes as:

$$X_1^T = \begin{bmatrix} 1+\delta^2 & 1 \end{bmatrix} \tag{2.17}$$

and

$$X_2^T = \begin{bmatrix} -\frac{1-\delta^2}{\mu} & 1 \end{bmatrix} \tag{2.18}$$

As shown in appendix, the modal strain and kinetic energy ratios are related as follows:

$$\beta_j = \Omega^2\left(\alpha_j^{1/2} \mp \mu^{1/2}\right)^2 \tag{2.19}$$

after proper replacements from equation (10), (13), and (14), and ignoring terms higher than δ squared, results in:

$$\beta_1 = \frac{\mu\delta^2}{(1+\mu)(1-2\delta^2)} \tag{2.20}$$

and

$$\beta_2 = \frac{1+\mu-2\delta^2}{\mu\delta^2} \tag{2.21}$$

Equation (2.20) reveals that in the fundamental mode the strain energy is associated almost exclusively with the distortion of the isolation system. On the contrary, equation (2.21) reveals that in the second mode the strain energy associated with the superstructure's distortion governs. It is to be noted that the product of the derived modal strain energy ratios is not exactly equal to one, as it should be per equation (A.8). The small error incurred, however, supports the good choice of the departing assumption for the mode shape, equation (2.7). We are now able to calculate the modal frequencies, in a normalized format, as:

$$\frac{\omega_1^2}{\omega_I^2} = (1+\mu)\frac{1+\beta_1}{1+\alpha_1} = 1 - \frac{\mu\delta^2}{1+\mu-2\delta^2} \tag{2.22}$$

and

$$\frac{\omega_2^2}{\omega_U^2} = \left(\frac{1}{\Omega^2}\right)\frac{1+\beta_2}{1+\alpha_2} = (1+\mu)\left(1+\frac{\mu\delta^2}{1+\mu-2\delta^2}\right) \tag{2.23}$$

Given the small values of δ squared, equation (2.22) shows that the fundamental frequency is slightly smaller that frequency of the rigid single-degree-of-freedom isolated superstructure. Likewise, equation (2.23) shows that the second frequency is approximately equal to the frequency of the fixed base superstructure times an amplification factor approximately equal to the square root of one plus the mass ratio. Since for any two-degree-of-freedom systems the following property applies:

$$\omega_1\omega_2 = \omega_U\omega_L \tag{2.24}$$

it is to be noted that depressing the fundamental frequency implies an consequential increase of the frequency of the second mode, as the right term of the latter equation is a constant. Equation (2.24) may also be written:

$$\left(\frac{\omega_1^2}{\omega_I^2}\right)\left(\frac{\omega_2^2}{\omega_U^2}\right) = 1+\mu \tag{2.25}$$

It may be easily verified that equation (2.22) and (2.23) satisfy equation (2.25). Per the theory in appendix, we can now write the participation factors as:

$$\gamma_1 = 1 - \frac{\mu\delta^2}{1+\mu-2\delta^2} = \frac{\omega_1^2}{\omega_I^2} \tag{2.26}$$

and

$$\gamma_2 = \frac{\mu\delta^2}{1+\mu-2\delta^2} = 1 - \frac{\omega_1^2}{\omega_I^2} \tag{2.27}$$

As it is well known, equation (2.26) and (2.27) indicate that the response of properly base isolated buildings is provided mainly by the fundamental mode. Further, in view of the modal normalization used, the summation of the participation factors must be equal to one. It can be immediately ascertained that equation (2.26) and (2.27) fulfill this condition. We verify then, that increasing the participation of the fundamental mode implies a consequential decrease of the participation of the second mode. It is interesting to note the relationship of the participation factors with the normalized frequency of the fundamental mode. The softer the isolation system, the closer the normalized frequency and the first participation factor are to one, which would be indicative of single-degree-of-freedom behavior.

3. CALCULATIONS OF MODAL DAMPING

Miranda [3], has shown that if coupling due to damping is ignored, a first order approximation to modal damping can be obtained as shown on equation (A.17). Let us recall that:

$$\xi_L = (1+\mu)^{1/2}\,\xi_I \tag{3.1}$$

where ξ_I is the damping ratio provided by the single-degree-of-freedom isolated superstructure, and ξ_L is the damping ratio associated with the lower mass of the system. Then, from equation (A.17), after the appropriate replacements and simplifications, we have the following approximation for the fundamental mode damping coefficient:

$$\xi_1 = \frac{\mu\delta^3}{(1+\mu)^{1/2}}\frac{\omega_U}{\omega_2}\xi_U + \frac{\omega_1^3}{\omega_I^3}\xi_I \cong \frac{\omega_1^3}{\omega_I^3}\xi_I \tag{3.2}$$

And for the second mode damping coefficient, the following approximation is found:

$$\xi_2 = (1+\mu)\frac{\omega_U}{\omega_2}\xi_U + \frac{\mu\delta}{(1+\mu)^{1/2}}\frac{\omega_I}{\omega_1}\xi_I \tag{3.3}$$

From equation (3.2) we see that in the fundamental mode, damping is provided almost exclusively by the isolation system, but that it is somewhat decreased with respect to the isolation system damping itself. Equation (3.3) indicates that the damping induced at the superstructure is amplified, while the damping provided by the single-degree-of-freedom isolated superstructure, *i.e.* by the isolation system, is depressed. Nevertheless, the contribution by the isolation system could still be significant for highly damped isolation systems. These observations are in parallel with the discussions made for the modal strain ratios in equation (2.20) and (2.21). Further exploration of the modal damping can be made if we define the modal *Level Damping Ratios* (*LRD*) per equation (A.20), and write it for seismically isolated buildings as:

$$(LDR)_j = \delta\beta_j\frac{\xi_U}{\xi_I} \tag{3.4}$$

The modal Level Damping Ratio, for the *j* mode, represents the ratio between the damping induced at the superstructure with respect to the damping induced in the single-degree-of-freedom isolated building, *e.i.* at the isolation system, while the systems vibrates freely. As seen from equation (3.4), the relative contribution to modal damping is directly proportional to the modal strain ratios. We can use equation (2.20) and (2.21) to calculate the level damping ratios, or conveniently use equation (3.2) and (3.3) to approximate them as:

$$(LDR)_1 = \frac{\mu\delta^3}{(1+\mu)}\frac{\omega_I^2}{\omega_1^2}\frac{\xi_U}{\xi_I} \tag{3.5}$$

and

$$(LDR)_2 = \frac{(1+\mu)}{\mu\delta}\frac{\omega_1^2}{\omega_I^2}\frac{\xi_U}{\xi_I} \tag{3.6}$$

We note from equation (3.5) that for the fundamental mode, damping is almost exclusively generated by the isolation system. On the contrary, equation (3.6) indicates that for the second mode, damping is generated mainly by the superstructure. However, the contribution of the isolators may still be significant, especially for highly damped systems.

From equation (A.19) we see that given a set of structural properties, the product of modal damping is a constant. This indicates that if the damping of the fundamental mode is increased, for example, a consequential decrease of the second mode damping ensues. For seismically isolated buildings, in view of the small values of β_1, and in view that β_2 is the dominant term, a simple approximation of equation (A.19) can be obtained as:

$$\xi_1\xi_2 = (1+\mu)^{1/2}\,\xi_U\xi_I + \frac{\mu\delta}{(1+\mu)^{1/2}}\xi_I^2 \tag{3.7}$$

To finalize, it can be shown that using the proper notation the structural dynamic characteristics derived above for isolated buildings idealized as two-degree-of-freedom systems; frequencies, modes, participation factors, and approximate damping, coincide with those derived by Kelly [2] through traditional eigenproblem solution procedures. Table I compares some results obtained using the expressions provided in this paper, with results using the corresponding equations provided by Kelly [2]. We see that to four decimal places, the results are virtually identical.

Table I. Comparison with results by Kelly [2], for $\mu = 5$, $\delta = 0.1$, $\xi_U = 0.02$, and $\xi_I = 0.1$.

Parameter	*Per Kelly [2]*	*Per this paper*
X_{U1}	1.0100	1.0100
X_{U2}	-0.1980	-0.1980
ω_1^2 / ω_I^2	0.9917	0.9916
ω_2^2 / ω_U^2	6.0500	6.0502
γ_1	0.9917	0.9916
γ_2	0.0083	0.0084
ξ_1	0.0988*	0.0988*
ξ_2	0.0691	0.0693

* Note: Damping of upper portion (superstructure) ignored.

4. SUMMARY AND CONCLUSIONS

Paralleling Kelly’s work [2], this paper studied seismically isolated buildings idealized as two-degree-of-freedom mechanical systems. Starting with a consideration of small drift for the fundamental mode of properly isolated buildings, the definition of the four modal energy ratios that determine all the structural dynamic properties is enabled. The calculation of the pertinent structural dynamic properties is performed then, using the modal kinetic and strain energy ratios relative to such systems. These properties include a first order approximation to modal damping. It can be concluded that the modal formulation leads to a compact presentation of the dynamic characteristics of seismically isolated buildings idealized as two-degree-of-freedom systems. It is also concluded that such formulation allows for a renewed visualization of the physical behavior of seismically isolated buildings. Finally, a simple numerical application indicates that the results obtained with the equations derived in this paper compare extremely well with the previous results provided by Kelly [2].

APPENDIX A

Let us consider the non-damped modes of the two-degree-of-freedom system depicted in Figure 1, assuming that it vibrates freely in its X_j mode. Let us then define the ratio α_j of the kinetic energy imparted to the upper mass with respect to the kinetic energy of the lower mass as:

$$\alpha_j = \frac{M_U X_{Uj}^2}{M_L X_{Lj}^2} \tag{A.1}$$

Let us similarly define β_j as the ratio of the strain energy stored in the upper spring with respect to the strain energy stored at the lower spring during free vibration. We write therefore:

$$\beta_j = \frac{K_U \left(X_{Uj} - X_{Lj} \right)^2}{K_L X_{Lj}^2} \tag{A.2}$$

If the modal components at the lower mass are normalized to one, from the last two equations the modal vectors can be written as:

$$X_j^T = \begin{bmatrix} \pm \dfrac{\alpha_j^{1/2}}{\mu^{1/2}} & 1 \end{bmatrix} \tag{A.3}$$

or alternatively

$$X_j^T = \begin{bmatrix} 1 \pm \dfrac{\beta_j^{1/2}}{\Omega \mu^{1/2}} & 1 \end{bmatrix} \tag{A.4}$$

In view that the modes are orthogonal with respect to the mass and stiffness matrices, the energy ratios fulfill the following equations:

$$\alpha_1 \alpha_2 = 1 \tag{A.5}$$

$$\Omega^2 = \frac{1}{1 + \sqrt{\mu \alpha_1} - \sqrt{\mu \alpha_2} - \mu} \tag{A.6}$$

$$\Omega = \frac{\left(\sqrt{\mu \beta_2} - \sqrt{\mu \beta_1} \right) + \sqrt{\left(\sqrt{\mu \beta_1} + \sqrt{\mu \beta_2} \right)^2 + 4}}{2 (1 + \mu)} \tag{A.7}$$

and

$$\beta_1 \beta_2 = 1 \tag{A.8}$$

From equation (A.3) and (A.4), it is seen that the energy ratios are related as follows:

$$\beta_j = \Omega^2 \left(\alpha_j^{1/2} \mp \mu^{1/2} \right)^2 \tag{A.9}$$

For a conservative system the maximum modal kinetic energy E_K is equal to the maximum strain energy E_S, and therefore in terms of the energies of the upper and lower components we can write for the jth mode:

$$\left(E_{KUj} + E_{KLj} \right)_{\max} = \left(E_{SUj} + E_{SLj} \right)_{\max} \tag{A.10}$$

or

$$\omega_j^2 M_L \left(1+\alpha_j\right) = K_L \left(1+\beta_j\right) \tag{A.11}$$

Hence, we obtain the modal frequencies as:

$$\left(\frac{\omega_j}{\omega_L}\right)^2 = \frac{1+\beta_j}{1+\alpha_j} \tag{A.12}$$

It may be shown that the modal frequencies comply with the following expressions:

$$\left(\frac{\omega_1}{\omega_L}\right)\left(\frac{\omega_2}{\omega_L}\right) = \Omega \tag{A.13}$$

and

$$\left(\frac{\omega_1}{\omega_L}\right)^2 + \left(\frac{\omega_2}{\omega_L}\right)^2 = 1+\left(1+\mu\right)\Omega^2 \tag{A.14}$$

The modal participation factors may be written as:

$$\gamma_1 = \frac{\sqrt{\alpha_2}+\sqrt{\mu}}{\sqrt{\alpha_1}+\sqrt{\alpha_2}} = \frac{\sqrt{\beta_2}}{\sqrt{\beta_1}+\sqrt{\beta_2}} \tag{A.15}$$

and

$$\gamma_2 = \frac{\sqrt{\alpha_1}-\sqrt{\mu}}{\sqrt{\alpha_1}+\sqrt{\alpha_2}} = \frac{\sqrt{\beta_1}}{\sqrt{\beta_1}+\sqrt{\beta_2}} \tag{A.16}$$

By equating the overall energy dissipation to the sum of the energy dissipated by each of the components, it can be shown that a first order approximation to the modal damping is provided by the following expression:

$$\xi_j = \frac{\beta_j}{\Omega\left(1+\beta_j\right)}\frac{\omega_j}{\omega_L}\xi_U + \frac{1}{\left(1+\beta_j\right)}\frac{\omega_j}{\omega_L}\xi_L \tag{A.17}$$

As shown by Miranda [3], this damping equation is exact only if the damping of the upper and lower components is in the following proportion:

$$\Omega = \frac{\xi_U}{\xi_L} \tag{A.18}$$

If such is the case, then the modal vectors derived above are able to de-couple the equations of motion for the two-degree-of-freedom system under consideration. We can manipulate equation (A.17) further, and in view of equation (A.13) we write the product of modal damping as:

$$\xi_1\xi_2 = \frac{\xi_U^2 + \Omega\left(\beta_1+\beta_2\right)\xi_U\xi_L + \Omega^2\xi_L^2}{\Omega\left(2+\beta_1+\beta_2\right)} \tag{A.19}$$

We now define the modal *Level Damping Ratio* (*LDR*), as the ratio between the damping generated at the upper level with respect to the damping generated at the lower level while the system vibrates in its *j*th mode. From (A.17), this is written as:

$$\left(LDR\right)_j = \frac{\beta_j}{\Omega}\frac{\xi_U}{\xi_L} \tag{A.20}$$

The product of the *Level Damping Ratios* is, therefore:

$$(LDR)_1 (LDR)_2 = \left(\frac{1}{\Omega} \frac{\xi_U}{\xi_L} \right)^2 \qquad \text{(A.21)}$$

Assuming that equation (A.18) is fulfilled, we have the particular forms:

$$(LDR)_j = \beta_j \qquad \text{(A.22)}$$

and

$$(LDR)_1 (LDR)_2 = 1 \qquad \text{(A.23)}$$

and

$$\xi_1 \xi_2 = \xi_U \xi_L \qquad \text{(A.24)}$$

DISCLAIMER

The findings, procedures, and opinions expressed in this paper are the sole responsibility of the author, and do not necessarily represent the practice of his employer, CH2M-Hill IDC.

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