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REMARKS ON WERNER AND HORODECKIS STATES AND OTHER RELATED TOPICS

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The spectral analysis of a family of states of bipartite quantum systems which are parametrized by two real parameters and which become Werner (respectively Horodeckis) states when one (or the other) of the parameters is dependent in a specific way on the other parameter is given. The channel fidelities of the quantum channels, which correspond to these states are calculated.

I. INTRODUCTION

It is well know that the Werner [1] and Horodeckis (isotropic) [2,3] states play an essential role in the quantum information theory of the finite dimensional quantum systems. In the present paper we define in an elementary way a family of quantum states parametrized by two real parameters which interpolate between Werner and Horodeckis states. These results where obtained before the more interesting and sophisticated results [4] of Vollbrecht and Werner where published. The Werner states are characterized by Werner parameter only and the Horodeckis states are characterized by the singlet fraction only. The Horodeckis proved [3] that the standard teleportation fidelity is a linear function of the singlet fraction. This result is a consequence of the duality between quantum channels and the quantum states with one reduced density matrix equal to the maximally mixed state [3,5-11]. In the present paper we remark that the Werner (and respectively Horodeckis) states are characterized by the complete determination of their Werner parameter (and respectively of their singlet fraction) by their singlet fraction (or respectively Werner parameter). Moreover, the corresponding dependences between Werner parameter and the singlet fraction multiplier with the dimension of the subsystems is the same linear function as in the case of the teleportation fidelity. Finally, we use the fact that in general the product between a Werner density matrix and a Horodeckis density matrix belongs to the family of the operators whose spectral analysis was already made in order to compute the fidelity [17-19] between a Werner state and a Horodeckis state.

2. THE WERNER AND HORODECKIS STATES

Let H be a $d-$dimensional Hilbert space. We denote by $End(H)$ the vector space of the linear operators on H. Let use denote by $\{u_1, u_2, ..., u_d\}$ a basis in the $d-$dimensional Hilbert space H. The operators E_{jk} are defined by:

$$E_{jk} u_l = \delta_{kl} u_j \tag{1.1}$$

If we consider the $d \times d$ $-$dimensional Hilbert space of a bipartite quantum system as the tensor product of the $d-$dimensional Hilbert spaces H that describe the pure states of each subsystem then we

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can take the basis $\{u_1,...,u_{d^2}\}$ in $H\otimes H$ in the following way $\{u_1\otimes u_1,...,u_d\otimes u_d\}$. With these conventions and defining the operators $V,P_+\in End(H\times H)$ in the following way:

$$V=\sum_{kl}E_{kl}\otimes E_{lk} \tag{1.2}$$

and

$$P_+=\frac{1}{d}\sum_{kl}E_{kl}\otimes E_{kl} \tag{1.3}$$

It is easily to give a proof that these two operators have the following properties:

$$\begin{aligned}&V^2=I\\&TrV=d\\&P_+^{\ 2}=P_+\\&TrP_+=1\\&VP_+=P_+V=P_+\end{aligned} \tag{1.4}$$

In fact, for any $A\otimes B\in End(H\times H)$ we have:

$$\begin{aligned}&Tr(A\otimes B)=Tr(A)Tr(B)\\&Tr(V(A\otimes B))=Tr(AB)\\&Tr(P_+(A\otimes B))=\frac{1}{d}T(AB^T)\end{aligned} \tag{1.5}$$

where the operator A^T is the transpose of the operator A in the above-defined basis. The partial transpose of the operators of the form $\sum A_k\otimes B_k\in End(H\times H)$ from is defined by:

$$\begin{aligned}&(\sum A_k\otimes B_k)^{T_1}=\sum A_k^{\ T}\otimes B_k\\&(\sum A_k\otimes B_k)^{T_2}=\sum A_k\otimes B_k^{\ T}\end{aligned} \tag{1.6}$$

Then we have:

$$\begin{aligned}&V^{T_1}=V^{T_2}=dP_+\\&P_+^{\ T_1}=P_+^{\ T_2}=\frac{1}{d}V\end{aligned} \tag{1.7}$$

We shall use also the relations $TrX^{T_1}Y=TrXY^{T_1}$ and $TrX^{T_2}Y=TrXY^{T_2}$ which are direct consequences of the definitions. For any density matrix σ on $H\otimes H$ we define two parameters: the singlet fraction $F(\sigma)=Tr\sigma P_+$ and the Werner parameter $\phi(\sigma)=Tr\sigma V$. Then we have:

$$\begin{aligned}&F(\sigma^{T_1})=F(\sigma^{T_2})=\frac{1}{d}\phi(\sigma)\\&\phi(\sigma^{T_1})=\phi(\sigma^{T_2})=dF(\sigma)\end{aligned} \tag{1.8}$$

There are two families of quantum states of bipartite quantum systems which are more carefully studied in the literature: the Werner ρ_W states and the Horodecki states ρ_H which are defined by their spectral decompositions:

$$\rho_W = \frac{\phi(\rho_W)+1}{d(d+1)}\frac{I+V}{2} + \frac{1-\phi(\rho_W)}{d(d-1)}\frac{I-V}{2} =$$
$$\frac{d-\phi(\rho_W)}{d(d^2-1)}I + \frac{d\phi(\rho_W)-1}{d(d^2-1)}V$$
$$\rho_H = F(\rho_H)P_+ + \frac{1-F(\rho_H)}{d^2-1}(I-P_+) = \tag{1.9}$$
$$\frac{1-F(\rho_H)}{d^2-1}I + \frac{d^2F(\rho_H)-1}{d^2-1}P_+$$

It follows immediately that:

$$\phi(\rho_H) = \frac{dF(\rho_H)+1}{d+1}$$
$$dF(\rho_W) = \frac{\phi(\rho_W)+1}{(d+1)} \tag{1.10}$$

The positivity of the eigenvalues is a consequence of the fact that the following restrictions on the parameters $\phi(\rho_W)$ and $F(\rho_H)$ are valid, as a consequence of their definitions:

$$-1 \le \phi(\rho_W) \le 1$$
$$0 \le F(\rho_H) \le 1 \tag{1.11}$$

We remark that:

$$(\rho_W)^{T_1} = (\rho_W)^{T_2} = \frac{d-\phi(\rho_W)}{d(d^2-1)}I + \frac{d\phi(\rho_W)-1}{d(d^2-1)}dP_+$$
$$(\rho_H)^{T_1} = (\rho_H)^{T_2} = \frac{1-F(\rho_H)}{d^2-1}I + \frac{d^2F(\rho_H)-1}{d^2-1}\frac{1}{d}V \tag{1.12}$$

Hence, the partial transposes of the Werner density matrices are Horodecki density matrices with $F((\rho_W)^{T_1}) = F((\rho_W)^{T_2}) = \frac{1}{d}\phi(\rho_W)$ and the partial transposes of the Horodecki density matrices are Werner density matrices with $\phi((\rho_H)^{T_1}) = \phi((\rho_H)^{T_2}) = dF(\rho_H)$. But, the positivity restrictions (1.9) impose new restrictions on these two parameters:

$$0 \le \phi(\rho_W) \le 1$$
$$0 \le F(\rho_H) \le \frac{1}{d} \tag{1.13}$$

We remark that these restrictions are the necessary and sufficient restrictions for the separability of Werner and respectively for Horodecki states. To any state σ we can associate two states $T_W(\sigma)$ and $T_H(\sigma)$ defined with the help of the spectral decompositions of the Werner and Horodecki density matrices respectively:

$$T_W(\sigma) = \frac{\phi(\sigma)+1}{d(d+1)} \frac{I+V}{2} + \frac{1-\phi(\sigma)}{d(d-1)} \frac{I-V}{2}$$
$$T_H(\sigma) = F(\sigma)P_+ + \frac{1-F(\sigma)}{d^2-1}(I-P_+) \qquad (1.14)$$

We remark that $T_W(\rho_W) = \rho_W$, $T_H(\rho_H) = \rho_H$. Also, we remark the invariance property with respect to these operations: $\phi(T_W(\sigma)) = \phi(\sigma), F(T_H(\sigma)) = F(\sigma)$. The more interesting relations where not remarked until now:

$$dF(T_W(\sigma)) = \frac{\phi(\sigma)+1}{(d+1)}$$
$$\phi(T_H(\sigma)) = \frac{dF(\sigma)+1}{d+1} \qquad (1.15)$$

3. QUANTUM CHANNELS ASSOCIATED WITH WERNER AND HORODECKI STATES.

There is an isomorphism between density matrices ρ acting on $H \otimes H$ which satisfy $Tr_2(\rho) = \frac{1}{d} I$ and the quantum channels Λ (i.e. completely positive and trace-preserving maps) acting on $End(H)$ given by:

$$\rho_\Lambda = (I \otimes \Lambda_\rho)P_+ \qquad (2.1)$$

For each channel Λ acting on $End(H)$ the channel fidelity $f(\Lambda)$ measures how close on average the input state $\sigma \in End(H)$ is to output state $\Lambda(\sigma) \in End(H)$. It is defined as:

$$f(\Lambda) = \int d\psi <\psi | \Lambda(|\psi><\psi|) |\psi> \qquad (2.2)$$

Where the integral is performed with respect to the uniform distribution $d\psi$ over all pure input states. For a state ρ acting on $H \otimes H$ we define

$$f(\rho) = f(\Lambda_{\Theta,\rho}) \qquad (2.3)$$

as the fidelity of the standard teleportation channel using ρ . For a channel Λ we define

$$F(\Lambda) = F(\rho_\Lambda) = F((I \otimes \Lambda)P_+) \qquad (2.4)$$

The quantum channels corresponding to the Werner and Horodecki density matrices are of the following form:

$$\Lambda_W(\sigma) = \frac{\phi(\rho_W)+1}{2(d+1)}[I+\sigma^T] + \frac{1-\phi(\rho_W)}{2(d-1)}[I-\sigma^T] \qquad (2.5)$$

and

$$\Lambda_H(\sigma) = \frac{F(\rho_H)d^2-1}{d^2-1}\sigma + \frac{(1-F(\rho_H))d}{d^2-1} I \qquad (2.6)$$

respectively. Then we obtain:

$$f(\Lambda_W) = <\psi|\Lambda_W(|\psi><\psi|)|\psi> = \frac{\phi(\rho_W)+1}{d+1} = dF(\rho_W)$$
$$f(\Lambda_H) = <\psi|\Lambda_H(|\psi><\psi|)|\psi> = \frac{F(\rho_H)d+1}{d+1} = \phi(\rho_H) \tag{2.6}$$

for the fidelities of the corresponding channels.

3. THE STATES WHICH INTERPOLATES BETWEEN THE WERNER AND HORODECKIS STATES

We define a two-parameter family of density matrices ρ_{WH} for a bipartite quantum system described by the Hilbert space $H \otimes H$ in the following way:

$$\rho_{WH} = xI + yV + zP_+ \tag{3.1}$$

where

$$xd^2 + yd + z = 1 \tag{3.2}$$

We shall consider in a first step only a tri-parametric family of operators of the following form:

$$O = xI + yV + zP_+ \tag{3.3}$$

with independent variables x, y, z. We remark that:

$$O^3 - (3x+y+z)O^2 + (3x^2 + 2xy + 2xz - y^2)O + (x+y+z)(x^2 - y^2)I = 0 \tag{3.4}$$

The equation:

$$\lambda^3 - (3x+y+z)\lambda^2 + (3x^2 + 2xy + 2xz - y^2)\lambda + (x+y+z)(x^2 - y^2)I = 0 \tag{3.5}$$

have the following roots:

$$\lambda_1 = x + y + z$$
$$\lambda_2 = x - y \tag{3.6}$$
$$\lambda_3 = x + y$$

from which it follows that:

$$x = \frac{1}{2}(\lambda_2 + \lambda_3)$$
$$y = \frac{1}{2}(-\lambda_2 + \lambda_3) \tag{3.7}$$
$$z = \lambda_1 - \lambda_3$$

In this way, we have obtained the spectral decomposition of any operator O:

$$O = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 \tag{3.8}$$

where P_1, P_2 and P_3 are the spectral projectors:

$$P_1 + P_2 + P_3 = I$$
$$P_k P_l = \delta_{kl} P_k \text{ , } k,l = 1,2,3 \tag{3.9}$$

and

$$P_1 = \frac{(O-\lambda_2 I)(O-\lambda_3 I)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}$$
$$P_2 = \frac{(O-\lambda_1 I)(O-\lambda_3 I)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} \qquad (3.10)$$
$$P_3 = \frac{(O-\lambda_1 I)(O-\lambda_2 I)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}$$

Using the fact that:

$$O^2 = (x^2+y^2)I + 2xyV + (2(x+y)+z)zP_+ \qquad (3.11)$$

we obtain:

$$P_1 = P_+$$
$$P_2 = \frac{1}{2}(I-V) \qquad (3.12)$$
$$P_3 = \frac{1}{2}(I+V-2P_+)$$

For a density matrix the restriction $Tr(O)=1$ gives $z = 1-(xd+y)d$ and

$$\lambda_1 = 1+x+y-d(dx+y)$$
$$\lambda_2 = x-y \qquad (3.13)$$
$$\lambda_3 = x+y$$

Hence:

$$\rho_{WH} = (x(1-d^2)+y(1-d)+1)P_+ + \frac{x-y}{2}(I-V) + \frac{x+y}{2}(I+V-2P_+) \qquad (3.14)$$

Let us denote by:

$$F(\rho_{WH}) = Tr(\rho_{WH}P_+) = x(1-d^2)+y(1-d)+1$$
$$\phi(\rho_{WH}) = Tr(\rho_{WH}V) = (x-y)d(1-d)+1 \qquad (3.15)$$

Then we have:

$$\lambda_1 = F(\rho_{WH})$$
$$\lambda_2 = \frac{1-\phi(\rho_{WH})}{d(d-1)} \qquad (3.16)$$
$$\lambda_3 = \frac{1+\phi(\rho_{WH})-2F(\rho_{WH})}{(d+2)(d-1)}$$

The positivity of the density matrix ρ_{WH} gives the following restrictions:

$$F(\rho_{WH}) \ge 0$$
$$\phi(\rho_{WH}) \le 1 \qquad (3.17)$$
$$1+F(\rho_{WH})-2\phi(\rho_{WH}) \ge 0$$

The density matrix ρ_{WH} parametrized by $\phi(\rho_{WH})$ and $F(\rho_{WH})$ have the following form:

$$\rho_{WH} = \frac{(1-dF(\rho_{WH})) + (d-\phi(\rho_{WH}))}{d(d+2)(d-1)} I + \frac{\phi(\rho_{WH})(d+1) - (1+dF(\rho_{WH}))}{d(d+2)(d-1)} V + \frac{d(d+1)F(\rho_{WH}) - (1+\phi(\rho_{WH}))}{(d+2)(d-1)} P_+ \tag{3.18}$$

In order to obtain the Werner density matrix the coefficient of the projector P_+ must vanish i.e.:

$$dF(\rho_{WH}) = \frac{1+\phi(\rho_{WH})}{d+1} \tag{3.19}$$

Analogously, in order to obtain the Horodecki density matrix the coefficient of the operator V must vanish, i.e.:

$$\phi(\rho_{WH}) = \frac{dF(\rho_{WH})+1}{d+1} \tag{3.20}$$

The partial transposes of the density matrices ρ_{WH} are given by:

$$(\rho_{WH})^{T_1} = (\rho_{WH})^{T_2} = \frac{(1-dF(\rho_{WH})) + (d-\phi(\rho_{WH}))}{d(d+2)(d-1)} I + \frac{\phi(\rho_{WH})(d+1) - (1+dF(\rho_{WH}))}{(d+2)(d-1)} P_+ + \frac{d(d+1)F(\rho_{WH}) - (1+\phi(\rho_{WH}))}{d(d+2)(d-1)} V \tag{3.21}$$

In order to have:

$$(\rho_{WH})^{T_1} = (\rho_{WH})^{T_2} = \rho_{WH} \tag{3.22}$$

the following relation must be valid:

$$dF(\rho_{WH}) = \phi(\rho_{WH}) \tag{3.23}$$

In this case the density matrices ρ_{WH} are given by:

$$\rho_{WH} = \frac{1+d-2dF(\rho_{WH})}{d(d+2)(d-1)} I + \frac{d^2F(\rho_{WH})-1}{d(d+2)(d-1)} V + \frac{d^2F(\rho_{WH})-1}{(d+2)(d-1)} P_+ \tag{3.24}$$

or by:

$$\rho_{WH} = \frac{1+d-2\phi(\rho_{WH})}{d(d+2)(d-1)} I + \frac{d\phi(\rho_{WH})-1}{d(d+2)(d-1)} V + \frac{d\phi(\rho_{WH})-1}{(d+2)(d-1)} P_+ \tag{3.25}$$

If we denote by $\tilde{x}, \tilde{y}, \tilde{z}$ and by $F((\rho_{WH})^{T_1}), \phi((\rho_{WH})^{T_1})$ the parameters of the partial transposes of the operator O and of the density matrix ρ_{WH} , then we have:

$$\begin{aligned} \tilde{x} &= x \\ d\tilde{y} &= z \\ \tilde{z} &= dy \end{aligned} \tag{3.26}$$

and

$$\begin{aligned}\phi((\rho_{WH})^{T_1}) &= dF(\rho_{WH})\\ dF((\rho_{WH})^{T_1}) &= \phi(\rho_{WH})\end{aligned} \tag{3.27}$$

Hence the products yz and Ff are invariants. For any density matrix, σ we define the operation $T_{WH}(\sigma)$ in the following way:

$$T_{WH}(\sigma) = F(\sigma)P_+ + \frac{1-\phi(\sigma)}{2d(d-1)}(I-V) + \frac{1+\phi(\sigma)-2F(\sigma)}{2(d+2)(d-1)}(I+V-2P_+) \tag{3.28}$$

Evidently, we have $T_{WH}(\rho_{WH}) = \rho_{WH}$. Also $F(T_{WH}(\sigma)) = F(\sigma)$ and $\phi(T_{WH}(\sigma)) = \phi(\sigma)$. We remark that for density matrices σ for which the relation $F(\sigma) = \frac{1+\phi(\sigma)}{d(d+1)}$ is valid we obtain that $T_{WH}(\sigma)$ is a Wener like state with the Werner parameter $\phi(\sigma)$. Also for density matrices σ for which the relation $\phi(\sigma) = \frac{1+dF(\sigma)}{d+1}$ is valid, we obtain that $T_{WH}(\sigma)$ is a Horodeckis like state with the singlet fraction $F(\sigma)$. The quantum channel Λ_{WH} associated with the state ρ_{WH} acts on $End(H)$ in the following way:

$$\Lambda_{WH}(\sigma) = dF(\rho_{WH})\sigma + \frac{1-\phi(\rho_{WH})}{2(d-1)}(I-\sigma^T) + \frac{d(1+\phi(\rho_{WH})-2F(\rho_{WH}))}{2(d+2)(d-1)}(I+\sigma^T-2\sigma) \tag{3.29}$$

The fidelity of this channel is given by:

$$f(\Lambda_{WH}) = <\psi|\Lambda_{WH}(|\psi><\psi|)|\psi> = \frac{F(\rho_{WH})d+\phi(\rho_{WH})+1}{d+2} \tag{3.30}$$

We remark that the this channel fidelity also interpolate between channel fidelities for Wener (2.6) and Horodeckis (2.6) states.

4. THE FIDELITY BETWEEN WERNER AND HORODECKIS STATES

The fidelity between two density matrices ρ_1 and ρ_2 is defined by the following formula:

$$\Phi(\rho_1,\rho_2) = (Tr\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}})^2 \tag{4.1}$$

In the following we shall compute the fidelity for the case $\rho_1 = \rho_H$ and $\rho_2 = \rho_W$. Because we have $\rho_1\rho_2 = \rho_2\rho_1$ then:

$$\Phi(\rho_1,\rho_2) = (Tr\sqrt{\rho_1\rho_2})^2 \tag{4.2}$$

and

$$O = \rho_H\rho_W = XI + YV + ZP_+ = \mu_1P_1 + \mu_2P_2 + \mu_3P_3 \tag{4.3}$$

Then

$$\sqrt{O} = \sqrt{\mu}_1P_1 + \sqrt{\mu_2}P_2 + \sqrt{\mu}_3P_3 \tag{4.4}$$

where

$$\mu_1 = \frac{F(\rho_H)(\phi(\rho_W)+1)}{d(d+1)}$$
$$\mu_2 = \frac{(1-\phi(\rho_W))(1-F(\rho_H))}{d(d^2-1)(d-1)} \tag{4.5}$$
$$\mu_3 = \frac{(\phi(\rho_W)+1)(1-F(\rho_H))}{d(d^2-1)(d+1)}$$

Then

$$Tr\sqrt{O} = \sqrt{\mu}_1 TrP_1 + \sqrt{\mu_2} TrP_2 + \sqrt{\mu}_3 TrP_3 \tag{4.6}$$

Hence

$$Tr\sqrt{O} = \sqrt{\mu_1} + \frac{d(d-1)}{2}\sqrt{\mu_2} + \frac{(d-1)(d+2)}{2}\sqrt{\mu_3} \tag{4.7}$$

Finally we have

$$\Phi(\rho_H,\rho_W) = \left(\sqrt{\frac{F(\rho_H)(1+\phi(\rho_W))}{d(d+1)}} + \frac{1}{2}\sqrt{\frac{(1-F(\rho_H))(1-\phi(\rho_W))}{d+1}} + \frac{d+2}{2(d+1)}\sqrt{\frac{(1-F(\rho_H))(1+\phi(\rho_W))(d-1)}{d}}\right)^2 \tag{4.8}$$

We remark that $\Phi\left(\frac{1}{d^2},\frac{1}{d}\right)=1$ as is expected because in this case $\rho_H = \rho_W = \frac{1}{d^2} I$. Also, we have $\Phi(1,-1)=0$ because in this case $\rho_H = P_+$, $\rho_W = \frac{1}{d(d-1)}(I-V)$.

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