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L.C.K-MANIFOLDS ON A TANGENT BUNDLE DETERMINED BY A RIEMANN, FINSLER OR LAGRANGE STRUCTURE

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We prove that any Riemann structure (*M, g*) determines a local conformal Kähler manifold on the tangent bundle (*TM*, π ,*M*). The Finsler structures and Lagrange structures have the same properties.

Key words : Lck-manifolds, Lagrange spaces

1. INTRODUCTION

As we know ([5], [7]), any Riemann structure R^n = (*M, g*) determines on the tangent bundle *TM* an almost Kähler structure (*TM*, $\overset{\circ}{\mathbb{G}}, \overset{\circ}{\mathbb{F}}$). We prove that the lift:

$$\left[\mathbb{G} = g_{ij}(x)dx^i \otimes dx^j + e^{2\sigma(x)} g_{ij}(x)\delta y^i \otimes \delta y^j\right]$$

of g to *TM* together with the correspondent almost complex structure:

$$\left[\mathbb{F}(x,y) = -e^{2\sigma(x)}\frac{\partial}{\partial y^i}\otimes dx^i + e^{2\sigma(x)}\frac{\delta}{\delta x^i}\otimes \delta y^i\right]$$

generate a local conformal almost Kähler structure (L.c.k)-structure. The same property holds for the Finsler spaces and Lagrange spaces.

2. THE ALMOST KÄHLER MANIFOLD DETERMINED BY A RIEMANN SPACE

Let $R^n = (M, g(x))$ be a Riemann space. The tangent bundle (*TM*, π ,*M*) has $(x^i, y^i), i = 1,...,n = \dim M$ as canonical local coordinates.

On the manifold *TM* there exist two distributions *N* and *V*, such that the following direct sum of linear spaces holds at any point $u \in TM$

$$[T_u TM = N_u \oplus V_u]$$

The horizontal distribution *N* has a local adapted basis $\frac{\delta}{\delta x^i}, (i = 1,...,n)$:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x,y)\frac{\partial}{\partial y^j} \tag{2.1}$$

where:

$$N_i^j = \gamma_{ih}^j(x) y^h \tag{2.2}$$

and $\gamma_{ih}^j(x)$ are the Christoffel symbols of the metric tensor $g_{ij}(x)$ of R^n.

The vertical distribution V is integrable, so that it has the adapted basis:

$$\left[\frac{\partial}{\partial y^i}, (i = 1, ..., n)\right]$$

The dual basis of $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ is given by $\left(dx^i, \delta y^i\right)$, where:

$$\delta y^i = dy^i + N_i^j(x, y) dx^j \tag{2.2'}$$

The Sasaki lift ([3], [4]) of the tensor metric $g_{ij}(x)$ to the manifold TM is defined by:

$$\overset{\circ}{\mathbb{G}}(x, y) = g_{ij}(x) dx^i \otimes dx^j + g_{ij}(x) \delta y^i \otimes \delta y^j \tag{2.3}$$

at any point $u = (x, y) \in TM$.

$\overset{\circ}{\mathbb{G}}$ is a Riemanian metric tensor on the manifold TM.

The horizontal distribution N determines a natural almost complex structure $\overset{\circ}{\mathbb{F}}$ on TM:

$$\overset{\circ}{\mathbb{F}}(x, y) = -\frac{\partial}{\partial y^i} \otimes dx^i + \frac{\delta}{\delta x^i} \otimes \delta y^i \tag{2.4}$$

It is a known fact that $\overset{\circ}{\mathbb{F}}$ is a complex structure if and only if the space R^n is locally flat.
Also, the following result is known ([5], [6], [7]).

Theorem 2.1.

1° The pair $\left(\overset{\circ}{\mathbb{G}}, \overset{\circ}{\mathbb{F}}\right)$ is an almost Hermitian structure on TM determined only by the Riemannian structure $g(x)$.

2° The almost symplectic structure associated to $\left(\overset{\circ}{\mathbb{G}}, \overset{\circ}{\mathbb{F}}\right)$ is given by:

$$\overset{\circ}{\theta} = g_{ij}(x) \delta y^i \wedge dx^j \tag{2.5}$$

The manifold $\left(TM, \overset{\circ}{\mathbb{G}}, \overset{\circ}{\mathbb{F}}\right)$ is an almost Kählerian space, determined only by a Riemannian structure space $g(x)$ i.e. we have:

$$d\overset{\circ}{\theta} = 0 \tag{2.6}$$

Also, we have:

Theorem 2.2. *The manifold $\left(TM, \overset{\circ}{\mathbb{G}}, \overset{\circ}{\mathbb{F}}\right)$ is a Kähler manifold if and only if the Riemann space $R^n = (M, g)$ is locally flat.*

3. L.C.K-MANIFOLD DETERMINED BY RIEMANNIAN SPACE R^n

Let us consider on the manifold *M* the conformal change:

$$g(x) \to e^{2\sigma(x)} g(x)$$

of the metric tensor *g*, where $\sigma(x)$ is a differentiable function, locally defined. We denote:

$$\overline{g}_{ij}(x) = e^{2\sigma(x)} g(x) \tag{3.1}$$

On the manifold *TM* we locally define the tensor field:

$$\mathbb{G}(x,y) = g_{ij}(x) dx^i \otimes dx^j + \overline{g}_{ij}(x) \delta y^i \otimes \delta y^i \tag{3.2}$$

where $\overline{g}_{ij}$ is from (3.1). The distributions *N* and *V* are orthogonal with respect to $\mathbb{G}$.

Consider, also, the tensor field:

$$\mathbb{F} = -e^{-2\sigma(x)} \frac{\partial}{\partial y^i} \otimes dx^i + e^{2\sigma(x)} \frac{\delta}{\delta x^i} \otimes \delta y^i \tag{3.3}$$

The following properties of $\mathbb{F}$ hold:

Theorem 3.1.

1º $\mathbb{F}$ is locally defined on the manifold TM.

2º $\mathbb{F}$ depends only on the conformal structure defined by the Riemann structure $g(x)$ on the base manifold M.

3º $\mathbb{F}$ is a tensor field of type (1,1).

4º $\mathbb{F}$ is an almost complex structure.

5º $\mathbb{F}$ is a complex structure if and only if:

a. The space $R^n = (M, g)$ is locally flat and

b. The function $\sigma(x)$ is constant.

The proof of this theorem is not difficult. Also, we can prove:

Theorem 3.2.

1º The pair $(\mathbb{G}, \mathbb{F})$ is an almost Hermitian structure locally determined on TM by the conformal structure $\overline{g}(x)$.

2º The almost symplectic structure $\overline{\theta}$ associated to $(\mathbb{G}, \mathbb{F})$ is:

$$\overline{\theta} = e^{2\sigma(x)} \theta$$

3º The following property holds:

$$d\overline{\theta} = 0, \left(\text{modulo} \overline{\theta}\right)$$

4º The manifold $(TM, \mathbb{G}, \mathbb{F})$ is locally conformal with an almost Kähler manifold.

If R^n is locally flat, then $(TM, \mathbb{G}, \mathbb{F})$ is a local conformal with a Kähler manifold.

4. L.C.K-MANIFOLD ON TM DETERMINED BY A FINSLER STRUCTURE

Let $F^n = (M, \mathbb{F}(x,y))$ be a Finsler manifold, having $\mathbb{F}(x,y)$ as fundamental function, and:

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}, \forall (x,y) \in \widetilde{TM} = TM \setminus \{0\} \tag{4.1}$$

as fundamental tensor field. We denote by *N* the Cartan nonlinear connection, with coefficients:

$$N^i_j(x,y) = \frac{1}{2}\frac{\partial}{\partial y^j}\left(y^j_{rs}(x,y)y^r y^s\right) \tag{4.2}$$

γ^i_{js} being the Christoffel symbols of the fundamental tensor field $g_{ij}(x,y)$ of the space F^n.

The direct decomposition $T_u TM = N_u \oplus V_u$ holds and an adapted basis to N_u and V_u, $\forall u \in \widetilde{TM} = TM \setminus \{0\}$ is given by $\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} (i = 1,...,n)$, where $\frac{\delta}{\delta x^i}$ are defined by (2.1) and (4.2). Its dual basis is $(dx^i, \delta y^i)$, the 1-forms δy^i being expressed in (2.2)', (4.2).

The Sasaki-Matsumoto ([5], [7]) lift of $g_{ij}(x,y)$ to $\widetilde{TM}$ is:

$$\overset{\vee}{\mathbb{G}}(x,y) = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)dy^i \otimes dy^j \tag{4.3}$$

and the almost complex structure $\overset{\vee}{\mathbb{F}}$ determined by the Cartan nonlinear connection is given by (2.4), (4.2).

The following result is known ([5], [7]):

Theorem 4.1.

1º The pair $\left(\overset{\vee}{\mathbb{G}}, \overset{\vee}{\mathbb{F}}\right)$ *is an almost Hermitian structure determined only by the fundamental function* $\mathbb{F}(x,y)$.

2º The almost symplectic structure associated to $\left(\overset{\vee}{\mathbb{G}}, \overset{\vee}{\mathbb{F}}\right)$ *is given by the 2-form:*

$$\overset{\vee}{\theta}(x,y) = g_{ij}(x,y)\delta y^i \wedge dx^j$$

3º The structure $\overset{\vee}{\theta}$ *is symplectic, i.e.,* $d\overset{\vee}{\theta} = 0$.

4º The manifold $\left(\widetilde{TM}, \overset{\vee}{\mathbb{G}}, \overset{\vee}{\mathbb{F}}\right)$ *is almost Kählerian.*

Now, let us consider the conformal transformation:

$$\overline{\mathbb{F}}(x,y) = e^{2\sigma(x)}\mathbb{F}(x,y), \tag{4.4}$$

of the fundamental function $\mathbb{F}(x,y)$, where $\sigma(x)$ is a local differentiable function on the base manifold *M*.

With respect to (4.4) we have the conformal transformation:

$$\overline{g}_{ij}(x,y) = e^{2\sigma(x)}g_{ij}(x,y), \tag{4.5}$$

of the fundamental tensor field. Now, we consider the following lift of $g_{ij}(x,y)$ and $\overline{g}_{ij}(x,y)$:

$$\overline{\mathbb{G}}(x,y)=g_{ij}(x,y)dx^i\otimes dx^j+e^{2\sigma(x)}g_{ij}(x,y)dy^i\otimes dy^j \tag{4.6}$$

and the following tensor field of (1,1) type:

$$\overline{\mathbb{F}}(x,y)=-e^{2\sigma(x)}\frac{\partial}{\partial y^i}+e^{2\sigma(x)}\frac{\delta}{\delta x^i}\otimes\delta y^i \tag{4.7}$$

The proof of the following theorem is not hard.

Theorem 4.2.

1º The pair $\left(\overline{\mathbb{G}},\overline{\mathbb{F}}\right)$ *is an almost Hermitian structure locally determined on the manifold* $\widetilde{TM}$ *by the fundamental functions* $\mathbb{F}(x,y)$ *and* $\overline{\mathbb{F}}(x,y)=e^{2\sigma(x)}\mathbb{F}(x,y)$.

2º The associated almost symplectic structure $\overline{\theta}$ *is:*

$$\overline{\theta}(x,y)=e^{2\sigma(x)}\check{\theta}(x,y)$$

3º The following property holds:

$$d\overline{\theta}=0\text{, (modulo }\overline{\theta}\text{)}$$

4º $\left(\widetilde{TM},\overline{\mathbb{G}},\overline{\mathbb{F}}\right)$ *is a (L.c.k)-manifold.*

Finally, we remark that the previous theory can be extended, step by step, to Lagrange spaces $L^n=\left(M,L(x,y)\right)$ ([4], [5]).

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