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# A PROPERTY OF A SYSTEM OF DETERMINANTAL EQUATIONS

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The aim of the paper is to show a property of the solutions of a system of determinantal equations. As an application we prove that an assertion is true if $n = 1,2,3$, but it does not hold if $n = 4$.

Properties of solutions of systems of nonlinear equations were investigated by many authors [1-8]. For a comprehensive survey on the systems of polynomial equations, see the relevant chapters in [1,2].

When the nonlinear equations are defined in terms of determinants, they are called determinantal equations (see [3] and [7]). The aim of the present paper is to exhibit a property of the solutions of a system of determinantal equations.

Let $A$ be a square matrix and consider the polynomial map $p(t) = \det(tI - A)$, $t \in \mathbf{C}$ and the set $\sigma(A) = \{ t \in \mathbf{C} : p(t) = 0\}$. The polynomial $p$ is called the characteristic polynomial of the matrix $A$ and $\sigma(A)$ is called the spectrum of $A$. The elements of $\sigma(A)$ are called eigenvalues of $A$.

If $q$ is a polynomial with complex coefficients

$$q(x) = c_0 x^m + c_1 x^{m-1} + \ldots + c_m ,$$

then we shall define the matrix

$$q(A) = c_0 A^m + c_1 A^{m-1} + \ldots + c_{m-1} A + c_m I$$

Denote by $M_n(K)$ the set of all square matrices of dimension $n$ whose entries belong to the field $K$.

**Theorem 1**. *Let $f_1, f_2, \ldots, f_n$ be monic polynomials with real coefficients whose roots are located on the unit circle and are different from $\pm 1$. Denote*

$$\mathsf{A} = \{A \in M_{2n}(\mathbf{R}) : \det[f_j(A)] = 0,\ \forall\ j \in \{1,2,\ldots,n\}\}\ \mathsf{B} = \{A \in M_{2n}(\mathbf{R}) : \det(A) = 1\}$$

*Then the following assertions are equivalent*:

1. $\gcd(f_j, f_k) = 1$ *for every* $j \neq k$
2. $\mathsf{A} \subset \mathsf{B}$

*Proof*. To prove 1⇒2 suppose that $\gcd(f_j, f_k) = 1$ for $j \neq k$ and let $A \in \mathsf{A}$. Note that if $\lambda \in \sigma(A)$ then $f_j(\lambda) \in \sigma(f_j(A))$. Since $0 = \det[f_1(A)] = \prod_{\lambda \in \sigma(A)} f_1(\lambda)$ it follows that there exists $\lambda_1 \in \sigma(A)$ such that $f_1(\lambda_1) = 0$. One can easily see that $|\lambda_1| = 1$ and $f_1(\bar{\lambda}_1) = 0$. Since $0 = \det[f_2(A)] = \prod_{\lambda \in \sigma(A)} f_2(\lambda)$ it follows that there exists $\lambda_2 \in \sigma(A)$ such that $|\lambda_2| = 1$ and $f_2(\lambda_2) = f_2(\bar{\lambda}_2) = 0$. By $\gcd(f_1, f_2) = 1$ it follows that $\lambda_1 \neq \lambda_2$. If we iterate the argument we obtain that for every $j \in \{2,3,\ldots,n\}$ there exists

$$\lambda_j \in \sigma(A) - \{\lambda_1, \lambda_2, \ldots, \lambda_{j-1}\} \text{ such that } |\lambda_j| = 1,\ f_j(\lambda_j) = f_j(\bar{\lambda}_j) = 0 .$$

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Consequently $\sigma(A)=\{\lambda_1,\overline{\lambda}_1,\lambda_2,\overline{\lambda}_2,...,\lambda_n,\overline{\lambda}_n\}$ whence

$$\det A=\prod_{\lambda\in\sigma(A)}\lambda=\lambda_1\overline{\lambda}_1\lambda_2\overline{\lambda}_2...\lambda_n\overline{\lambda}_n=|\lambda_1\lambda_2...\lambda_n|^2=1$$

To prove 2 ⇒1, suppose that $\gcd(f_1, f_2)=d$ is a nonconstant polynomial. Let $\lambda_2, \lambda_3,\dots,\lambda_n$ be complex numbers such that $d(\lambda_2)=0$ and $f_j(\lambda_j)=0,\ j=3,4,...,n$. One can easily see that $\lambda_j\in \mathbf{C}-\mathbf{R}$, $|\lambda_j|=1$ for $j=2,3,...,n$. Let $t_j\in\mathbf{R}$ be such that $\lambda_j=\cos t_j+i\sin t_j$ for $j=2,3,...,n$ and consider the matrices

$$A_j=\begin{pmatrix}\cos t_j & \sin t_j\\ -\sin t_j & \cos t_j\end{pmatrix}\qquad j=2,3,...,n$$

By $d(\lambda_2)=0$ it follows that $d(A_2)=0$ whence $f_1(A_2)=f_2(A_2)=0$. By $f_j(\lambda_j)=0$ it follows that $f_j(A_j)=0,\ \ j=3,4,\dots,n$. Consider the $2n\times 2n$ matrix

$$A=\begin{pmatrix}A_1 & 0 & 0 & ... & 0\\ 0 & A_2 & 0 & ... & 0\\ 0 & 0 & A_3 & ... & 0\\ ... & ... & ... & ... & ...\\ 0 & 0 & 0 & ... & A_n\end{pmatrix}$$

with $A_1=0$. We shall use the shorthand $A=\operatorname{diag}(A_1, A_2,...,A_n)$. Note that:

$f_1(A)=\operatorname{diag}((f_1(0),0,\ f_1(A_3),...,\ f_1(A_n)),$
$f_2(A)=\operatorname{diag}((f_2(0),0,\ f_2(A_3),...,\ f_2(A_n)),$
$f_3(A)=\operatorname{diag}((f_3(0),\ f_3(A_2),0,...,\ f_3(A_n)),$
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$f_n(A)=\operatorname{diag}((f_n(0),\ f_n(A_2),\ f_n(A_3),...,\ f_n(A_{n-1}),0)$.

Thus $\det[f_j(A)]=0$ for every $j=1,2,\dots,n$. Note that $\det A=\det A_1\det A_2\dots\det A_n=0$. Consequently $A\in \mathsf{A}-\mathsf{B}$. This contradicts assertion 2.

**Corollary 2.** *Let $f_1, f_2,..., f_n$ be irreducible monic polynomials with real coefficients whose roots are located on the unit circle and are different from $\pm 1$. Suppose that $A\in M_{2n}(\mathbf{R})$ is a matrix such that* $\det[f_j(A)]=0$ *for every $j=1,2,\dots,n$. If $f_1, f_2,..., f_n$ are distinct then* $\det A=1$.

**Corollary 3.** *Let $m_j\ge 2,\ j=1,2,\dots,n$ be natural numbers such that* $\gcd(m_j, m_k)=1$ for every $j\ne k$. *Denote $f_j(t)=1+t+t^2+...+t^{m_j-1};\ j=1,2,\dots,n$. If $A\in M_{2n}(\mathbf{R})$ is a matrix such that* $\det[f_j(A)]=0$, $j=1,2,\dots,n$ *then* $\det A=1$.

*Proof*. One can easily see that all the roots of the polynomials $f_j,\ j=1,2,\dots,n$ are located on the unit circle and are different from $\pm 1$. Since $\gcd(t^j-1, t^k-1)=t^{\gcd(j,k)}-1$ it follows that $\gcd(f_j, f_k)=1$ for $j\ne k$. The conclusion of the corrolary follows at once from Theorem 1.

Consider the following statement:

**Statement A5.** *Let $n\in\mathbf{N}^*$ and $A, B\in M_{2n}(\mathbf{R})$. If $AB=BA$ and* $(*)\sum_{k=1}^{n}\det\left[A^{2k}+A^{2k-1}B+...+AB^{2k-1}+B^{2k}\right]=0$ *then* $\det A=\det B$.

One can easily note that:

**Remark 4.** *The conclusion of statement* $A5$ *holds if the matrices A and B are singular.*

**Remark 5.** *Since* $\det\left[A^{2k} + A^{2k-1}B + \ldots + AB^{2k-1} + B^{2k}\right] \geq 0$ *for every* $k \geq 1$ *it follows that the equation* (*) *implies that* $\det\left[A^{2k} + A^{2k-1}B + \ldots + AB^{2k-1} + B^{2k}\right] = 0$ *for every* $k = 1,2,\ldots,n$

**Remark 6.** *Statement* $A5$ *is equivalent to*

**Statement A5'.** *If* $f_j(t) = 1 + t + t^2 + \ldots + t^{2j}$ , $j = 1,2,\ldots,n$ *and* $C \in M_{2n}(\mathbf{R})$ *is a matrix such that* $\det\left[f_j(C)\right] = 0$ *for every* $j = 1, 2,\ldots,n$ *then* $\det(C) = 1$.

**Remark 7.** *From Theorem* 1 *we see that the conclusion of statement* $A5'$ *holds if* $n = 1,2,3$ *but it does not hold if* $n = 4$. *In the latter case we see that* $\gcd(f_1, f_4) = f_1 \neq \text{const}$.

## REFERENCES


1. COHEN A. M., CUYPERS H., and STERK H., editors. *Some Tapas of Computer Algebra*. Springer-Verlag, 1999.
2. COX D., LITTLE J. and O'Shea D., *Ideals, Varieties, Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra.* UTM., Springer-Verlag, New York, 1992.
3. GOROKH, O.V.; WERNER, F., *On the solution of determinantal systems of linear inequalities.*, J. Optimization **35**, No.4, 301-316, 1995.
4. SOMMESE A. J., VERSCHELDE J., *Numerical homotopies to compute generic points on positive dimensional algebraic sets.,* J. Complexity **16**, No.3, 572-602, 2000.
5. SOMMESE A. J., VERSCHELDE J. and WAMPLER C. W., *Numerical decomposition of the solution sets of polynomial systems into irreducible components*, SIAM J. Numer. Anal. **38**, No.6, 2022-2046, 2001.
6. SOMMESE A. J., VERSCHELDE J. and WAMPLER C. W., *Symmetric functions applied to decomposing solution sets of polynomial systems.,* SIAM J. Numer. Anal., **40**, No.6, 2026-2046, 2002.
7. VERSCHELDE J., *Polynomial homotopies for dense, sparse and determinantal systems*, MSRI Preprint 1999-041, www.msri.org/publications/preprints/ online/1999-041.html
8. VERSCHELDE J., *Toric Newton method for polynomial homotopies*. J. Symb. Comput. **29**, No.4-5, 777-793, 2000.


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