PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 5, Number 1/2004, pp.000 - 000

A NEW GEOMETRIZATION OF TIME DEPENDENT LAGRANGIANS

Camelia FRIGIOIU*, Magdolna KIRKOVITS**

*University "Dunãrea de Jos", Faculty of Sciences, Department of Mathematics, Galaþi, Romania
** University of Debrecen, Hungary
Corresponding author: Camelia Frigioiu, e-mail: cfrigioiu@ugal.ro

Two different geometrizations of time dependent Lagrangians were proposed by M. Anastasiei, [1], and by M. Anastasiei and H. Kawaguchi, [2], using the projections $\mathbf{R}\times TM \to \mathbf{R}\times M$ and $\mathbf{R}\times TM \to M$, respectively. In this paper we propose a geometrization of time dependent Lagrangians using the geometry of the manifold $\mathbf{R}\times TM$ itself. This is more compatible with the one resulting from a general problem of local equivalence treated by B. Lackey, [3].

Key words: Time dependent Lagrangian; Vector bundle; Linear connection; Manifold.

1.INTRODUCTION

A time dependent Lagrangian is a smooth, real function defined on the manifold $\mathbf{R}\times TM$, for M a smooth manifold and TM its tangent bundle. The manifold $\mathbf{R}\times TM$ can be naturally projected on $\mathbf{R}\times M$ or on M and thus it appears as the total space of two different vector bundles. Accordingly, two different geometrizations of time dependent Lagrangians were proposed by M. Anastasiei([1], see also Ch. 13 in [4]) and M. Anastasiei and H. Kawaguchi, [2], using the general thery of the vector bundles developped in a book by R. Miron and M. Anastasiei, [4].

In this paper we propose a geometrization of time dependent Lagrangians using the geometry of $\mathbf{R}\times TM$ without any additional structure of vector bundle.

As we shall work in local coordinates, for an easier control of the geometrical meaning of the objects on the manifold $\mathbf{R}\times TM$ we shall decompose its tangent bundle in three subbundles and we shall use only the objects adapted to this decomposition. We show that any regular time dependent Lagrangian produces such decomposition.

Our geometrization is more compatible with the one resulting from a general problem of local equivalence treated by B. Lackey, [3], using the Cartan method of equivalence.

The plan of the paper is as follows. In §2 we discuss the geometry of $\mathbf{R}\times TM$ and we describe the decomposition of its tangent bundle just mentioned. In §3 we define a time dependent Lagrangian L and its extremals as curves on M which are solutions of the Euler-Lagrange equations. A formula for variation of the energy of L along extremals is given. Next we consider regular time dependent Lagrangians and we see how they influences the geometry of $\mathbf{R}\times TM$. A semispray derived from a regular time dependent Lagrangian is proposed. Section 4 is devoted to an almost metrical contact structure on $\mathbf{R}\times TM$ completely determined by a time dependent Lagrangian. In the last section we consider linear connections which are in some sense compatible with a time dependent Lagrangian and we show the existence of a family of such connections. Choosing one, we find that its torsions and curvatures provide invariants found by B. Lackey, [3], when studying the local equivalence of the pairs consisting in a semispray and a metric.

Recommended by Radu MIRON, Member of the Romanian Academy

2.THE MANIFOLD *R´TM*

Let M be a smooth i.e. $C^{¥'}$ manifold of finite dimension n and TM its tangent bundle.

Let $\left(x^i\right), i, j, k, ... = 1,2,..., n$ be local coordinates on M and $\left(x^i, y^i\right)$ be the local coordinates on TM. Then $\left(t, x^i, y^i\right)$ are local coordinates on $\boldsymbol{R}\times TM$.As in Analytical Mechanics we consider the time t as an absolute invariant and thus a change of coordinates $(t, x^i, y^i) \to \left(t, \tilde{x}^i, \tilde{y}^i\right)$ will be taken in the form

$$\tilde{t} = t, \tilde{x}^i = \tilde{x}^i(x^1, x^2, ..., x^n), \ \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j}(x) y^j, \ \text{rank}\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) = n. \tag{2.1}$$

Under the natural basis (2.1) $\left(\frac{\partial}{\partial t} := \partial_t, \ \frac{\partial}{\partial x^i} := \partial_i, \ \frac{\partial}{\partial y^i} := \dot{\partial}_i\right)$ of local vector fields on $\boldsymbol{R}\times TM$ has the law of transformation:

$$\tilde{\partial}_t = \partial_t, \ \dot{\partial}_i = \partial_i(\tilde{x}^j)\dot{\tilde{\partial}}_j, \ \partial_i = \partial_i(\tilde{y}^j)\dot{\tilde{\partial}}_j + \partial_i(\tilde{x}^j)\tilde{\partial}_j \tag{2.2}$$

while its dual basis $\left(\mathrm{d}t, \mathrm{d}x^i, \mathrm{d}y^i\right)$ transforms as:

$$\mathrm{d}\tilde{t} = \mathrm{d}t, \ \mathrm{d}\tilde{x}^i = \partial_j(\tilde{x}^i)\mathrm{d}x^j, \ \mathrm{d}\tilde{y}^i = \partial_j(\tilde{x}^i)\mathrm{d}y^j + \partial_j(\tilde{y}^i)\mathrm{d}x^j. \tag{2.3}$$

We put $\delta_i = \partial_i - N_i^j\dot{\partial}_j - N_i^0\partial_t$, for undetermined functions $\left(N_i^j\right)$ and $\left(N_i^0\right)$ on domains of local charts on $\boldsymbol{R}\times TM$ and we require that (δ_i) transform by rule

$$\delta_i = \partial_i(\tilde{x}^j)\tilde{\delta}_j. \tag{2.4}$$

It follows that the functions $\left(\tilde{N}_h^k\right)$ and $\left(\tilde{N}_h^0\right)$ defining $\tilde{\delta}_j$ and $\left(N_i^j\right)$, $\left(N_i^0\right)$ have to satisfy

$$\tilde{N}_h^k\partial_i(\tilde{x}^h) = \partial_j(\tilde{x}^k)N_i^j - \partial_i(\tilde{y}^k), \ N_j^0 = \partial_j(\tilde{x}^k)\tilde{N}_k^0. \tag{2.5}$$

We notice that the functions $\left(N_i^j\right)$ behave like the coefficients of a nonlinear connection on M and the functions $\left(N_i^0\right)$ behave like the components of a covector on M. One can take $N_j^0 = 0$, while one cannot $N_j^i = 0$.

We set H=span(δ_i), V_0=span(∂_t), V=span$(\dot{\partial}_i)$ and we have a decomposition of the tangent space of $\boldsymbol{R}\times TM$ into the following direct sum

$$T_u\left(R\times TM\right) = H \oplus V_0 \oplus V, \quad u \in R\times TM. \tag{2.6}$$

We shall report any geometric object on $\boldsymbol{R}´TM$ the decomposition (2.6).Then their components will change like those of corresponding objects on M although they depend on t and (y^i).

The local basis $\left(\partial_t, \delta_i, \dot{\partial}_i\right)$ is adapted to the decomposition (2.6).Its dual is $\left(\delta t, dx^i, \delta y^i\right)$, where

$$\delta y^i = dy^i + N_k^i dx^k, \ \delta t = dt + N_k^0 dx^k. \tag{2.7}$$

We define a tensor field J of type (1,1) on $\boldsymbol{R}´TM$ as

$$J(\partial_t) = 0, \ J(\dot{\partial}_i) = 0, \ J(\partial_i) = J(\delta_i) = \dot{\partial}_i. \tag{2.8}$$

This definition is working because of (2.2) and one easily gets

$$J^2 = 0. \tag{2.9}$$

We say that J defines an almost tangent structure on $\boldsymbol{R}\times TM$. This is integrable since as it is easy to check, the Nijenhuis tensor field of J vanishes.

The tensor field P of type (1,1) on $\boldsymbol{R}\times TM$ given by

$$P(\partial_t) = -\partial_t,\ P(\delta_i) = \delta_i,\ P(\dot{\partial}_i) = -\dot{\partial}_i \tag{2.10}$$

is well defined and $P^2=I$ (Kronecher's tensor field), i.e. P defines an almost product structure on $\boldsymbol{R}\times TM$.

3.TIME DEPENDENT LAGRANGIANS

A time dependent Lagrangian is a smooth scalar function $L: R\times TM \to R$, $(t,x,y)\to L(t,x,y)$.

Let $c:[0,1]\to M$ be a smooth curve on M given a local chart by $x^i=x^i(t)$, $t\in[0,1]$.Its tangent vector field $\dot{c}:[0,1]\to TM$ is represented as $\left(x^i(t), \dot{x}^i(t) = \dfrac{dx^i}{dt}\right)$ and the map $t\to\left(t, x^i(t), \dot{x}^i(t)\right)$ is a smooth curve on $\boldsymbol{R}\times TM$ while $I(c)=\int_0^1 L(t, x(t), \dot{x}(t))dt$ is called the action integral. The following variational problem appears: find among the set of all curves from M with the same endpoints those which afford extremal values for I. The first variational formula for I shows that the curves which we are looking for, called extremals of L, have to be solutions of the Euler-Lagrange equations

$$E_i(L) := \frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = 0. \tag{3.1}$$

The function

$$E_L(t,x,y) = y^i \frac{\partial L}{\partial y^i}(t,x,y) - L(t,x,y), \tag{3.2}$$

is called the energy of the Lagrangian L.

We may also consider the function

$$E_L^{'}(t,x,y) = t\frac{\partial L}{\partial t} + y^i \frac{\partial L}{\partial y^i}(t,x,y) - L(t,x,y) \tag{3.2'}$$

as a more specific energy of L.

A direct calculation gives:

Lemma 3.1. *Along any curve from M, the following formulae hold*:

$$\frac{\mathrm{d}E_L}{\mathrm{d}t} = -\frac{\mathrm{d}x^i}{\mathrm{d}t}E_i(L) - \frac{\partial L}{\partial t}, \tag{3.3}$$

$$\frac{\mathrm{d}E_L^{'}}{\mathrm{d}t} = -\frac{\mathrm{d}x^i}{\mathrm{d}t}E_i(L) + t\frac{\mathrm{d}(\partial_t L)}{\mathrm{d}t}. \tag{3.3'}$$

Thus we have:

Theorem 3.1. (i) *The variation of energy E_L along an extremal of L is given by*

$$\frac{\mathrm{d}E_L}{\mathrm{d}t} = -\frac{\partial L}{\partial t}, \tag{3.4}$$

(ii) *The variation of energy $E_L^{'}$ along an extremal of L is given by*

$$\frac{\mathrm{d}E_L^{'}}{\mathrm{d}t} = t\frac{\mathrm{d}(\partial_t L)}{\mathrm{d}t}. \tag{3.4'}$$

Corollary 3.1. *If the Lagrangian L does not depend on time , then* $E_L = E_L^{'}$ *and* E_L *is conserved along the extremals of L.*

Definition 3.1. A time dependent Lagrangian is called *regular* if the matrix with the entries

$$g_{ij}(t,x,y) = \dot{\partial}_i \dot{\partial}_j (L^2/2) \tag{3.5}$$

is of rank n.

The condition from Definition 3.1. does not depend on the local coordinates since when passing to the coordinates $(t, \tilde{x}, \tilde{y})$ ew have

$$g_{ij}(t,x,y) = \partial_i(\tilde{x}^k)\partial_j(\tilde{x}^h)\tilde{g}_{kh}(t,\tilde{x},\tilde{y}). \tag{3.6}$$

We say that the functions $(g_{ij}(t,x,y))$ define a d-tensor field of type (0,2). This is symmetric.Here "d" is for "distinguished".We note that $(E_i(L))$ defines a d-covector field, because of

$$E_i(L) = \partial_i(\tilde{x}^k)\tilde{E}_k(L). \tag{3.7}$$

Theorem 3.2. *For a regular time dependent Lagrangian, the Euler-Lagrange equations take the form*

$$\frac{d^2x^i}{dt^2} + 2\Gamma^i(t,x,\dot{x}) = 0, \tag{3.8}$$

with

$$\begin{aligned} \Gamma^i &= G^i + H_0^i, \\ 4G^i &= g^{ij}[(\dot{\partial}_j \partial_k L)y^k - \partial_j L], \\ 4H_0^i &= g^{ij}\dot{\partial}_j(\partial_t L), \quad \dot{x}^i = y^i. \end{aligned} \tag{3.8'}$$

Proof. If we expand in (3.1) the derivative with respect to t, we obtain

$$2g_{ij}\frac{\mathrm{d}^2x^j}{\mathrm{d}t^2} + \left(\frac{\partial^2 L}{\partial \dot{x}^i \partial x^j}\dot{x}^j - \frac{\partial L}{\partial x^i}\right) + \frac{\partial^2 L}{\partial \dot{x}^i \partial t} = 0.$$

Transvecting this equation by g^{hi}, the entries of the inverse of the matrix (g_{ij}) yields (3.8) with the notation (3.8)' .

Theorem 3.3. *The functions* $N_j^i(t,x,y) = \dot{\partial}_j \Gamma^i(t,x,y)$ *verify (2.5) when a change of coordinates (2.1) is performed.*

Proof. Under a change of coordinates (2.1), L and $\frac{\partial L}{\partial t}$ remain invariant and the other partial derivativ of L undergo the following changes:

$$\frac{\partial L}{\partial x^i} = \frac{\partial \overline{x}^k}{\partial x^i}\frac{\partial L}{\partial \tilde{x}^k} + \frac{\partial \tilde{y}^k}{\partial x^i}\frac{\partial L}{\partial \tilde{y}^k},$$

$$\frac{\partial^2 L}{\partial y^i \partial t} = \frac{\partial \tilde{x}^k}{\partial x^i}\frac{\partial^2 L}{\partial \tilde{y}^k \partial t},$$

$$\frac{\partial^2 L}{\partial y^i \partial x^k} = \frac{\partial^2 L}{\partial \tilde{y}^j \partial \tilde{x}^h}\frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial \tilde{x}^h}{\partial x^k} + \frac{\partial L}{\partial \tilde{y}^j}\frac{\partial^2 \tilde{x}^j}{\partial x^i \partial x^k} + 2\tilde{g}_{jh}\frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial \tilde{y}^h}{\partial x^h}.$$

We use these in the equation : $4g_{ij}G^j = \frac{\partial^2 L}{\partial y^i \partial x^k}y^k - \frac{\partial L}{\partial x^i}$

and on account of rank$\left(\tilde{g}_{ij}\right)$=rank$(g_{ij})=n$, after a small calculation we obtain

$$\frac{\partial \tilde{x}^k}{\partial x^i} G^i = \tilde{G}^k + \frac{1}{2}\frac{\partial^2 \tilde{x}^k}{\partial x^i \partial x^j} y^i y^j , \quad \frac{\partial \tilde{x}^k}{\partial x^i} H_0^i = \tilde{H}_0^k . \tag{3.9}$$

Adding these two equations and differentiating the result with respect to $\left(y^i\right)$, yield

$$\frac{\partial \tilde{x}^k}{\partial x^i} \dot{\partial}_j (G^i + N_0^i) = \partial_i (\tilde{x}^h\} \frac{\partial}{\partial \tilde{y}^h} (\tilde{G}^k + \tilde{H}_0^k) + 2\partial_i (\tilde{y}^k),$$

that is $\left(\dot{\partial}_j \Gamma^i\right)$ verify (2.5) q.e.d.

Concluding, if a regular time dependent Lagrangian is given , on $\boldsymbol{R}\times TM$ we have a decomposition (2.6) with δ_i constructed using $N_j^i = \dot{\partial}_j \Gamma^i$ and $\left(N_i^0\right)$ arbitrary; $\left(N_i^0\right)$ can be taken 0 or $\left(\dot{g}_{ik} H_0^k\right) = \left(\frac{1}{4}\frac{\partial^2 L}{\partial y^i \partial t}\right)$ since both verify (2.5). Hence we have two different decompositions (2.6) which will produce different geometries for $\boldsymbol{R}\times TM$ endowed with the function L. The choice $N_i^0 = 0$ creates more compatibility with the treatment of L from the viewpoint of a local equivalence problem.

In that follows we take $N_i^0 = 0$ and $N_j^i = \dot{\partial}_j \Gamma^i$. Let us consider on $\boldsymbol{R}\times TM$ the vector field

$$S_L = \frac{\partial}{\partial t} + y^i \frac{\partial}{\partial x^i} - 2\Gamma^i \frac{\partial}{\partial y^i} . \tag{3.10}$$

This is well-defined since it takes the form

$$S_L = \partial_t + y^i \delta_i + (N_j^i y^j - 2\Gamma^i)\dot{\partial}_i , \tag{3.10'}$$

and it is easy to see, using (3.9), that the functions

$$\varepsilon^i = 2\Gamma^i - N_j^i y^j , \tag{3.11}$$

define a d-vector field.

The vector field S_L is completely determined by L and it is called the semispray of L.

The integral curves of S_L are solutions of the following system of differential equations:

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = 1, \quad \frac{\mathrm{d}x^i}{\mathrm{d}\tau} = y^i , \quad \frac{\mathrm{d}y^i}{\mathrm{d}\tau} = -2\Gamma^i (t, x, y). \tag{3.12}$$

We may take $t = \hat{o}$and then (3.12) implies (3.8). Thus if we project the curve $\left(t, x(t), y(t)\right)$ from $\boldsymbol{R}\times TM$ on to the curve $c{:}t$® $x(t)$ in M , we just have obtained that the integral curves of S_L are projected on the extremals of L. Conservely, if we take a solution $(x^i(t))$ of (3.8) and we lift it to the curve $\left(t, x^i(t), \dot{x}^i(t)\right)$ on $\boldsymbol{R}\times TM$, we obtain that the lift curve is an integral curve of S_L.

We notice that S_L verifies

$$JS_L = C , \tag{3.13}$$

where C= $y^i \dot{\partial}_i$ is the Liouville vector field on $\boldsymbol{R}\times TM$.

We can extend the notion of semispray saying that a vector field S on $\boldsymbol{R}\times TM$ is a semispray if it satisfies (3.13). Then S will be of the form

$$S = a(t, x, y)\partial_t + y^i \partial_i + S^i \dot{\partial}_i , \tag{3.14}$$

for every function a and for every (S^i) such that the functions $(S^i + N_j^i y^j)$ define a d-vector field.

4. AN ALMOST CONTACT STRUCTURE ON R ´TM

Definition 4.1.[4} The pair $RL^n=(M,L)$ with L a time dependent Lagrangian, such that the quadratic form $Q=g_{ij}(t,x,y)\xi^i\xi^j, (\xi^i)\in \mathbf{R}^n$, is of constant curvature is called a rheonomic space.

We associate to L a metrical structure on $\boldsymbol{R}\times TM$,

$$G(t,x,y) = a\delta t\otimes\delta t + g_{ij}(t,x,y)dx^i\otimes dx^j + g_{ij}(t,x,y)\delta y^i\otimes\delta y^j \tag{4.1}$$

where a is a nowhere zero function on $\boldsymbol{R}\times TM$.

This is a Riemannian structure if the quadratic form Q is positive definite. Otherwise, it is a pseudo-Riemannian structure.

We define a tensor field F of type (1,1) by

$$F(\partial_t) = 0,\ F(\delta_i) = \dot{\partial}_i,\ \ F(\dot{\partial}_i) = -\delta_i. \tag{4.2}$$

This is clearly of rank$F = 2n$. The kernel of it is the distribution V_0 on $\boldsymbol{R}\times TM$.

Theorem 4.1. *The triple* $(F,\partial_t,\delta t)$ *is an almost contact structure on* $\boldsymbol{R}\times TM$*, that is, we have*

$$F^2 = -I + \partial_t\otimes\delta t,\ \delta t(\partial_t) = 1,\ \ F(\partial_t) = 0. \tag{4.3}$$

Proof. It suffices to check that $F^2X = -X + \delta t(X)\partial_t$ for X=$\partial_t, \delta_i, \dot{\partial}_i$. And this is easy using (4.2).

Theorem 4.2. *We have*

$$G(FX,FY) = G(X,Y) - \delta t(X)\delta t(Y),\ \mathrm{G}(\partial_t, X) = \delta t(X), \tag{4.4}$$

for every vector fieds X,Y on $\boldsymbol{R}\times TM$ *if and only if* a *º1*.

Proof. Easy checkings in adapted basis, using (4.1) and (4.2).

Corollary 4.1. *The ensemble* $(F,\partial_t,\delta t, G)$ *for a º1 is a metrical almost contact structure on* ***R*** *´TM*.

The almost contact structure $(F,\partial_t,\delta t)$ is said to be normal if $N_F + d(\delta t)\otimes\partial_t = 0$, where

$$N_F(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y],$$

is the Ninjenhuis tensor field associated to F.

A direct calculation in adapted basis gives

Theorem 4.3. *The almost contact structure* $(F,\partial_t,\delta t)$ *is normal if and only if*

1) $R^i_{jk} := \delta_k N^i_j - \delta_j N^i_k = 0,$

2) $\dfrac{\partial N^i_j}{\partial t} = 0,\ \dfrac{\partial N^0_i}{\partial t} = 0.$

Remark 4.1. The condition 1) above is equivalent to the integrability of the distribution H on $\boldsymbol{R}\times TM$.

5. NORMAL D-CONNECTIONS ON *R´TM*.

Now we are interested to find linear connections on ***R*** *´TM* which to be compatible in a way or another with a time dependent Lagrangian L.

Let D be a linear connection on ***R*** *´TM*. It will be known if one knows $D_{\partial_t}X$, $D_{\delta_i}X$, $D_{\dot{\partial}_i}X$ for every vector field X on ***R*** *´TM*.. As ***R*** *´TM* has three complementary distributions it is quite natural to look for connections which preserves these distributions.

Definition 5.1. A linear connection on D is called a d-connection if it preserves by parallelism the distributions H, V_0, V on ***R*** *´TM*.

In the adapted basis $(\partial_t, \delta_i, \dot{\partial}_i)$ a d-connection D takes the form

$$D_{\partial_t}\partial_t = f(t,x,y)\partial_t, D_{\partial_t}\delta_j = L^i_j\delta_i, \ D_{\partial_t}\dot{\partial}_j = \tilde{L}^i_j\dot{\partial}_i,$$
$$D_{\delta_j}\partial_t = M_j\partial_t, D_{\delta_j}\delta_k = \Gamma^i_{jk}\delta_i, \ D_{\delta_{jt}}\dot{\partial}_k = \tilde{\Gamma}^i_{jk}\dot{\partial}_i, \tag{5.1}$$
$$D_{\dot{\partial}_j}\partial_t = \tilde{M}_j\partial_t, D_{\dot{\partial}_j}\delta_k = \tilde{C}^i_{kj}\delta_i, \ D_{\dot{\partial}_j}\dot{\partial}_k = C^i_{kj}\dot{\partial}_i,$$

where f is a function, $L^i_j, \tilde{L}^i_j, M_j, \tilde{M}_j, C^i_{kj}$ and $\tilde{C}^i_{kj}$ are d-tensor fields and $\Gamma^i_{kj}, \tilde{\Gamma}^i_{kj}$ behave like the connection coefficients on M although they depend on t,x and y.

On using (5.1) it is easy to prove

Theorem 5.1. *A linear connection on* $\boldsymbol{R}\times TM$ *is a d-connection if and only if*

$$D_X P = 0, \text{ for every vector field } X \text{ on } \boldsymbol{R}\times TM. \tag{5.2}$$

Recall that on $\boldsymbol{R}\times TM$ we have also the tensor field J with the property $J^2=0$.

Definition 5.2. A d-linear connection will be called a normal d-connection, shortly an nd-connection if

$$D_X J=0, \text{ for every vector field } X \text{ on } \boldsymbol{R}\,´TM. \tag{5.3}$$

A direct calculation based on (5.1) gives:

Theorem 5.2. *A d-connection on* $\boldsymbol{R}\times TM$ *is an nd-connection if only if*

$$\tilde{L}^i_j = L^i_j, \ \tilde{\Gamma}^i_{kj} = \Gamma^i_{kj}, \ \tilde{C}^i_{kj} = C^i_{kj}. \tag{5.4}$$

A linear connection D on $\boldsymbol{R}\times TM$ is called metrical if $DG=0$. A direct calculation in adapted basis and taking into account (5.4) leads to:

Theorem 5.3. *An nd-connection on* $\boldsymbol{R}\times TM$*.is metrical if and only if*

$$\partial_t g_{ij} = L^k_i g_{kj} + L^k_j g_{ki},$$
$$\delta_i g_{jk} = \Gamma^h_{ji} g_{hk} + \Gamma^h_{ki} g_{hj}, \tag{5.5}$$
$$\dot{\partial}_i g_{jk} = C^h_{ji} g_{hk} + C^h_{ki} g_{hj}.$$

$$f = \frac{1}{2}\frac{\partial a}{\partial t}, \ M_i = \frac{1}{2}\delta_i a, \ \tilde{M}_i = \frac{1}{2}\dot{\partial}_i a. \tag{5.6}$$

Remark 5.1. When a is a constant different from zero (5.6) shows that $D(\partial_t) = 0$, that is, the distribution V_0 is absolute parallel. Another good candidate for a is L itself, assuming it is nowhere zero.

Then $f = \frac{1}{2}\frac{\partial L}{\partial t}, \ M_i = \frac{1}{2}\delta_i L, \tilde{M}_i = \frac{1}{2}\dot{\partial}_i L$.

From the previous considerations we extract

Corollary 5.1. (i) *A metrical nd-connection D is characterized by*

$$DP=0, DJ=0, DG=0, \tag{5.7}$$

and it also satisfies

$$DF=0. \tag{5.7)'}$$

(ii) *A metrical nd-connection D is completely determined by the functions* $\left(N^i_j\right), \left(N^i_0\right)$ *and a as well as by the triple* $D\Gamma = \left(L^i_j, \Gamma^i_{kj}, C^i_{kj}\right)$ *such that* (5.5) *holds.*

Let D be a metrical nd-connection. If one estimates the torsion of DG on the adapted basis $(\partial_t, \partial_i, \dot{\partial}_i)$ it comes out that it is completely determined by the following d-tensor fields on $\boldsymbol{R}\times TM$:

$$L^i_j, \partial_t N^i_j, M_j, \tilde{M}_j$$

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}, \; R^i_{jk} = \delta_k N^i_j - \delta_j N^i_k, C^i_{jk}, \tag{5.8}$$

$$P^i_{kj} = \dot{\partial}_j N^i_k - \Gamma^i_{kj}, \; S^i_{jk} = C^i_{jk} - C^i_{kj}.$$

Similarly, one finds that the curvature of D is completely determined by the following d-tensor fields on $\boldsymbol{R}\times TM$:

$$Q_i = \partial_t M_i - \delta_i f + \partial_t\left(N^k_i\right)\tilde{M}_k, \; \tilde{Q}_i = \partial_t \tilde{M}_i - \dot{\partial}_i f,$$

$$M^i_{kj} = \partial_t \Gamma^i_{kj} + \Gamma^s_{kj} L^i_s - \delta_j L^i_k - L^s_k \Gamma^i_{sj} + \partial_t (N^s_j) C^i_{sk}$$

$$H^i_{kj} = \dot{\partial}_t C^i_{kj} + C^s_{kj} L^i_s - \dot{\partial}_j L^i_k - L^s_k C^i_{sj}$$

$$R^i_{hjk} = \delta_k \Gamma^i_{hj} - \delta_j \Gamma^i_{hk} + \Gamma^s_{hj}\Gamma^i_{sk} - \Gamma^s_{hk}\Gamma^i_{sj} + C^i_{hs} R^s_{jk} \tag{5.9}$$

$$P^i_{hjk} = \dot{\partial}_k \Gamma^i_{hj} - C^i_{hk|j} + C^i_{hs} P^s_{jk}$$

$$S^i_{hjk} = \dot{\partial}_k C^i_{hj} - \dot{\partial}_j C^i_{hk} + C^s_{hj} C^i_{sk} - C^s_{hk} C^i_{sj},$$

where $_{|j}$ means the covariant derivative constructed with $\left(\Gamma^i_{kh}\right)$, called h-covariant derivative.

Besides h-covariant derivative we may consider a v-covariant derivative defined by C^i_{jk}, for instance $z^i\big|_k = \dot{\partial}_k z^i + C^i_{hk} z^h$, as well as a 0-covariant derivative, for instance $z^i\big|_0 = \partial_t z^i + L^i_k z^k$.

With these definitions and notations, (5.5) takes the form

$$g_{ij}\big|_0 = 0, \; g_{ij|k} = 0, \; g_{ij}\big|_k = 0. \tag{5.5)'$$

The tensorial equations (5.5) in the unknown L^j_i, Γ^i_{kj}, C^i_{kj} can be solved in certain conditions. Thus if we take $T^i_{jk} = 0 \Leftrightarrow \Gamma^i_{kj} = \Gamma^i_{jk}$, the second equation (5.5) has the unique solution

$$\Gamma^i_{kj} = \frac{1}{2} g^{ih}(\delta_j g_{hk} + \delta_k g_{hj} - \delta_h g_{kj}). \tag{5.10}$$

Indeed, if in this equation we circularly permute the indices i,j,k and substract one equation thus obtained from the sum of the other two, taking into account the hypothesis $T^i_{jk} = 0$, we get (5.10).

Similarly, we see that under the hypothesis $S^i_{jk} = 0$, the third equation (5.5) has the unique solution

$$C^i_{jk} = \frac{1}{2} g^{ih}(\dot{\partial}_j g_{hk} + \dot{\partial}_k g_{hj} - \dot{\partial}_h g_{kj}) = \frac{1}{2} g^{ih}\dot{\partial}_j g_{hk}. \tag{5.11}$$

By using the Obata operators associated to $\left(g_{ij}\right)$ it comes out that the first equation (5.5) has the general solution

$$L^i_j = \frac{1}{2} g^{ih}\partial_t g_{jh} + \frac{1}{2}\left(\delta^k_j \delta^i_h - g_{jh} g^{ki}\right) X^h_k \tag{5.12}$$

where $\left(X^h_k\right)$ is an arbitrary d-tensor field on $\boldsymbol{R}\times TM$. Thus we have a family of metrical nd-connections. If we prescribe $\left(X^h_k\right)$ or take it zero select one from this set which will be determined by $\left(g_{ij}\right)$ only. Moreover, if we take $X_k^h = 0$ and *a º1*, we obtain a metrical *nd*-connection which depends on L only. A choice of a as a function of L is also possible.

If the dependence of time is removed from L, the geometry that we just developped reduces to the geometry of a Lagrange space as it is presented, for instance, in [4]. This forces us to choose $X_k^h = 0$ and *a º1* in order that the metrical *nd*-connection given by (5.10) and (5.11) with $N_j^i = \dot{\partial}_j G^i$ to be the standard metrical connection from Lagrange geometry.

REFERENCES

1. ANASTASIEI, M., *The geometry of time dependent Lagrangians*, Math. Comput. Modellinng **20**, pp.67-81,1994;
2. ANASTASIE, M., KAWAGUCHI, H., *A geometrical theory of time dependent Lagrangians*, I. Non-linear connections, II.N-connections, Tensor N.S. **48** , pp.273-293, 1989;
3. LACKEY, B., *Metric equivalence of path spaces*, Nonlinear Studies, **7** , 241-250, 2000;
4. MIRON, R., ANASTASIEI, M., *Lagrange Spaces: Theory and applications,* Kluwer Acad. Publ. FTPH 59, 1994.

Received September, 5, 2003