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A CLASSICAL LIMIT

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Sets of Kronecker products of finite dimensional irreducible representations of Lie algebras of types B_n, C_n and D_n, possessing the following properties are pointed out : a) For a given set, the highest weight of one of the factors in the Kronecker product is a multiple $m\Lambda_i$ of a given fundamental weight with $m=$ positive and integer ; b) For a given set, the Clebsch-Gordan decomposition is the same for any m and depends only on m. Only sets with Clebsch-Gordan series lengths three and four have been considered. For a given set of this type, the limit for $m \to \infty$ of the minimal polynomial equations satisfied by the Hannabuss-operator associated with the Kroneckker products lead to equations (of degree three and four) for equivariant operators K, defined previously ; these equations are equivalent to classical tensorial identities

I. INTRODUCTION

It has been long been known that realizations of a semi-simple Lie algebra L satisfy specific polynomial identities, both for classical realizations (i.e. for Poisson bracket realizations) as well as for quantum realizations (i.e. for linear representations). In previous works (Refs 1-3) we pointed out that such identities results by equating to zero well defined tensors in the symmetric algebra $S(L)$ of L and in the enveloping algebra $U(L)$, respectively. We therefore used the term "tensorial identities". These tensors are transforming under symmetric subrepresentations of Kronecker powers of the adjoint representation of L. All second-degree symmetric tensors in $S(L)$ and in $U(L)$ have been determined (Refs. 2,3). As is well known [4] the last ones result from the first ones by symmetrization with respect to order in the products. For the quantum realizations of the semisimple Lie algebras of types A_n, B_n, C_n and D_n, all finite-dimensional linear representations which satisfy the "tensorial identities" of degree two have been determined [3]. By using, in addition, a method due to Hannabuss [5], who relates the tensorial identity satisfied by a representation ρ_Λ of L with highest weight Λ to its Kronecker product with other representations of L, it has been proved that second-degree identities for finite-dimensional representations of classical semi-simple Lie algebras are related to specific Kronecker products $\rho_\Lambda \otimes \rho_\Omega$, where ρ_Λ is a minuscule representation [6] and ρ_Ω (its "partner") is a representation whose highest weight Ω is of the form $\Omega = m\Lambda_i$, where Λ_i is a well-defined fundamental weight and m is an arbitrary positive integer. (These results are summarized in Table I of Ref. 7.) The $m-$dependence of the "partner weights" $\Omega = m\Lambda_i$, $(m=1,2,...)$ suggests a possibility to derive a classical limit for the identities associated with the product $\rho_\Lambda \otimes \rho_\Omega$, by taking the limit $m \to \infty$. In the limit $m \to \infty$ the (second-degree) identities satisfied by the Hannabuss operators $O_{\Lambda,\Omega}$ associated with the pairs of highest weights listed in Table I (of the Ref. 7) go into second- degree identities for the operators K (defined in Refs. 2, 8), which are associated with ρ_Λ and with a Poisson bracket realization. The existence of this classical limit is based on the possibility to associate Poisson bracket realizations of a Lie algebra L to group orbits through the highest weights of finite dimensional representations of L. The aim of the present paper is to point out identities of degrees higher than two satisfied by linear representations. To do that we need to point out sets of Kronecker products possessing the following properties : a) All terms of a given set of Kronecker products have Clebsch-Gordan (CG) decompositions of equal length. b) For a given set, the CG decomposition is the same for any m and depends only on m. These properties ensure that the minimal polynomials satisfied by the Hannabuss operators have the same expression and the same dependence on m. Sets of Kronecker products with these properties can be constructed by using an extension of a theorem due to Feingold [9]. (cf. also Ref. 10). We have identified in this way, for the classical Lie algebras of types B_n, C_n and D_n, sets of Kronecker products whose CG series are of lengths three and four. By taking the limit, for $m \to \infty$, of the minimal polynomials satisfied by the Hannabuss operators associated with these Kronecker products we obtained identities satiosfied by the corresponding $K-$ operators. The classical limits of the minimal polynomials polynomials associated with each set of products are summarized in the last section.

2. INVARIANT AND EQUIVARIANT OPERATORS

a) The Hannabus operator.

Let ρ_{Λ} be a finite-dimensional representation with highest weight Λ of the semisimple $n-$ dimensional Lie algebra L. We denote by e_k, $k = 1,...,n$ a basis in L and by e^k, $k = 1,...,n$ a dual basis in L with respect to the Cartan-Killing bilinear form : $(e_k, e^j) = \delta_k{}^j$. Then the second-order Casimir operator of the representation ρ_{Λ} is defined by:

$$C_2(\Lambda) = \sum_{k=1}^{n} \rho_{\Lambda}(e_k) \otimes \rho_{\Lambda}(e^k) \tag{2.1}$$

and the Hannabus operator of two representations ρ_{Λ} and ρ_{Ω} is defined by [5] (cf. also Ref. 11) :

$$\mathrm{O}_{\Lambda,\Omega} = \sum_{k=1}^{n} \rho_{\Lambda}(e_k) \otimes \rho_{\Omega}(e^k) \tag{2.2}$$

The operator $\mathrm{O}_{\Lambda,\Omega}$ has been defined by Hannabus [5]; cf. also Ref. [11].

Properties of $\mathrm{O}_{\Lambda,\Omega}$.

1. The operator $\mathrm{O}_{\Lambda,\Omega}$ is invariant under adjoint action.
2. $\mathrm{O}_{\Lambda,\Omega}$ can be expressed in terms of Casimir operators of the representations $\rho_{\Lambda} \otimes \rho_{\Omega}$, ρ_{Λ} and ρ_{Ω}.
3. $\mathrm{O}_{\Lambda,\Omega}$ commutes $\rho_{\Lambda} \otimes \rho_{\Omega}$.
4. Let :

$$\rho_{\Lambda} \otimes \rho_{\Omega} = \bigoplus_{\Xi \in CG(\Lambda,\Omega)} \rho_{\Xi} \tag{2.3}$$

be the CG decomposition of product $\rho_{\Lambda} \otimes \rho_{\Omega}$ (whose set of weghts is denoted by $CG(\Lambda,\Omega)$). The minimal polynomial satisfied by the operator $\mathrm{O}_{\Lambda,\Omega}$ has the expression:

$$\prod_{\Xi\in CG(\Lambda,\Omega)}\left[O_{\Lambda,\Omega}-\frac{1}{2}((\Xi+2\delta,\Xi)-(\Lambda+2\delta,\Lambda)-(\Omega+2\delta,\Omega))\right] \tag{2.4}$$

where 2δ is the sum of the positive roots of the Lie algebra L. The degree of Eq. (2.4) is clearly equal with the length of the CG series of $\rho_\Lambda\otimes\rho_\Omega$. The matrix elements of the characteristic equation (obtained by equating to zero the minimal polynomial) taken between states belonging to one of the representations (ρ_Λ, say) provide identities satisfied by the representation ρ_Ω. The invariance property of the operator $\mathrm{O}_{\Lambda,\Omega}$ implies that the matrix elements of the minimal polynomial are tensors.

a) The moment-like mapping.

Let us replace in Eq. (2.2) one of the quantum realization by a classical one, e.g. write instead of the generator $\rho_\Omega(e_j)$ of the linear representation ρ_Ω the generator $f_{e_j}(m)$ of the Poisson bracket realization on a symplectic G-manifold M, ($e_j\in L,\ m\in M$). We obtain, in this way, a mapping:

$$K\ :\ M\to EndV_\Lambda \tag{2.5}$$

defined by :

$$K(m)=\sum_{j=1}^{n} f_{e_j}(m)\,\rho_\Lambda\ (e^j),\ (m\in M) \tag{2.6}$$

Assuming that $S_\Lambda(g)$ is the representation of G ($L=Lie(G)$) acting in V_Λ, the following property of the operator K can be proved :

$$K(g.m)=S_\Lambda(g^{-1})K(m)S_\Lambda(g)\quad (g\in G) \tag{2.7}$$

i.e. the mapping K is equivariant. An immediate consequence is that any polynomial $P(K)$ is equivariant. Polynomial relations satisfied by the operator K are equivalent to sets of polynomial relations satisfied by the generators of the Poisson bracket realization.

3. THE CLASSICAL LIMIT

To transform Eq. (2.4) satisfied by $\mathrm{O}_{\Lambda,\Omega}$ into an equation for an operator K of type (2.6) we remind that the Poisson brackets realizations on the co-adjoint orbit through a highest weight Λ is generated by the covariant symbols, i.e. :

$$f_x(\xi)=f_x(Ad(g)^{\bullet}\Lambda)=f_{Ad(g)x}(\Lambda)=\langle v_\Lambda|\rho_\Lambda(Ad(g)x)|v_\Lambda\rangle \tag{3.1}$$

where $|v_\Lambda>$ denotes the highest weight vector of the representation ρ_Λ. Reminding that:

$$\rho_\Lambda(Ad^*(g)x)=S^{-1}{}_\Lambda(g)\rho_\Lambda(x)S_\Lambda(g) \tag{3.2}$$

and that:

$$S_\Lambda(g)\otimes S_\Omega(g)O_{\Lambda,\Omega}S_\Lambda(g^{-1})\otimes S_\Omega(g^{-1})=O_{\Lambda,\Omega} \tag{3.3}$$

we define the operator $K_{\Lambda,\Omega}$ associated with the Hannabuss operator $\mathrm{O}_{\Lambda,\Omega}$ by:

$$K_{\Lambda,\Omega}(\Lambda)=\sum_{i=1}^{n} f_{e_j}(\Lambda)\rho_\Omega(e^j)=<v_\Lambda\,|\,O_{\Lambda,\Omega}\,|\,v_\Lambda> \tag{3.4}$$

whence, with:

$$\begin{aligned}
&\xi = Ad^*(g)\Lambda\\
&K_{\Lambda,\Omega}(\xi)=\sum_{i=1}^{n} f_{e_j}(Ad^*(g)\Lambda)\rho_\Omega(e^j)=<v_\Lambda\,|\,(S_\Lambda(g)\otimes I)O_{\Lambda,\Omega}(S_\Lambda(g^{-1})\otimes I)\,|\,v_\Lambda>=\\
&<v_\Lambda\,|\,(I\otimes S_\Omega(g^{-1}))O_{\Lambda,\Omega}(I\otimes S_\Omega(g))\,|\,v_\Lambda>=S_\Omega(g^{-1})<v_\Lambda\,|\,O_{\Lambda,\Omega}\,|\,v_\Lambda>S_\Omega(g)=\\
&S_\Omega(g^{-1})K_{\Lambda,\Omega}(\Lambda)S_\Omega(g)
\end{aligned} \tag{3.5}$$

and the equivariance property of the operator K is proved. As:

$$<v_{m\Lambda}\,|\,\rho_{m\Lambda}(x)\,|\,v_{m\Lambda}>=m\Lambda(x) \tag{3.6}$$

we have:

$$K_{m\Lambda,\Omega}(\xi)=mK_{\Lambda,\Omega}(\xi) \tag{3.7}$$

As proved in Ref. 12, the principal part of:

$$<v_{m\Lambda}\,|\,[O_{m\Lambda,\Omega}]^q\,|\,v_{m\Lambda}> \tag{3.8}$$

behaves like $m^q K_{\Lambda,\Omega}$. In particular, for $q=2$, second-degree equations for $O_{m\Lambda,\Omega}$ become second-degree equations for operator denoted $K_\Omega(\xi)=K_{\Lambda,\Omega}(\xi)$.

For the Hannabuss operators $O_{m\Lambda,\Omega}$ associated with the products $\rho_{m\Lambda}\otimes\rho_\Omega$ (with the CG series of length 2) the coefficients of the minimal polynomials:

$$(O_{m\Lambda,\Omega}-k_1 I)(O_{m\Lambda,\Omega}-k_2 I)=0 \tag{3.9}$$

where given in Ref. [8]. Hence the coefficients $b=\lim\limits_{m\to\infty}\frac{-1}{m}(k_1+k_2)$ and $c=\lim\limits_{m\to\infty}\frac{1}{m^2}k_1k_2$ of the second-degree equation satisfied by K_Ω can be obtained directly [8].

4. FEINGOLD'S THEOREM

For the sake of transparency, starting with the present section we shall use for a representation of higest weight Λ the notation (Λ) instead of ρ_Λ. To obtain the classical limits for relations of degree three and four satisfied by the Hannabuss operator we proceeded in the following way : We found out the sets of Kronnecker products of type $(\Lambda)\otimes(m\Lambda_i)$ $(m=1,2,...)$ with the following properties:

a)The products $(\Lambda)\otimes(m\Lambda_i)$ $(m=1,2,...)$ admits CG decompositions of equal lengths;

b) The terms of these decompositions have the same expression (in particular, the same m-dependence).

To construct sets of Kronecker products with these properties a generalization of a theorem due to Feingold [9] is of great help. To state it we introduce the following notations: $\alpha_1,\alpha_2,...,\alpha_n$ are the simple rootsc of the complex semi-simple Lie algebra L of rank n, $\alpha=\frac{2\alpha^v}{(\alpha,\alpha)}$ is the coroot of the root α, Λ^+ alis the set of dominant weights corresponding to $\alpha_1,\alpha_2,...,\alpha_n$, $\Lambda_1,\Lambda_2,...,\Lambda_n$ are the fundamental weights in Λ^+, defined by :

$$<\Lambda,\alpha_j{}^v>=\delta_{ij} \tag{4.1}$$

Simple roots and fundamental weights are related by:

$$\alpha_i=\sum_{j=1}^{n}<\alpha_i,\alpha_j{}^v>\Lambda_j \tag{4.2}$$

where $<\alpha_i,\alpha_j{}^v>$ are elements of the Cartan matrix of L . For the Lie algebras A_n,D_n,E_6,E_7,E_8 the lengths of all roots are equal. For the Lie algebras C_n,B_n,F_4,G_2 fthe systems of roots divides in two subsystems: long roots and short roots. For each of these last Lie algebras, two roots are dominant weights : the highest long root α_{hl} and the highest short root α_{hs}. Their expressions and as well as those of the corresponding coroots are displayed in Ref. [6]. With these prerequisites we shall now formulate Feingold's theorem in the form stated in Ref. [10].

THEOREM. Let L be a semi-simple Lie algebra of rank n. Let Λ,Ω cbe dominasnt weights of L, $(\Lambda,\Omega\in\Lambda^+)$, and let (Λ), (Ω) be the corresponding finite-dimensional irreducible representations. Assume that for the long (short) simple roots α_i of L we have:

$$<\Omega,\alpha_i{}^v>\;\geq\;<\Lambda,\alpha_{hl(hs)}{}^v> \tag{4.3}$$

Then, if:

$$(\Lambda)\otimes(\Omega)=\bigoplus_{\Gamma\in\Lambda^+}m_\Gamma(\Gamma) \tag{4.4}$$

(m_Γ is the multiplicity of the representation (Γ)) we have also:

$$(\Lambda)\otimes(\Omega+\Lambda_i)=\bigoplus_{\Gamma\in\Lambda^+}m_\Gamma(\Gamma+\Lambda_i) \tag{4.5}$$

(In Eq. (4.3) $\alpha_{hl(hs)}{}^v$ has the following meaning $\alpha_{hl(hs)}{}^v=\alpha_{hl}{}^v(\alpha_{hs}{}^v)$ if α_i is long (short). Using the same notation as in the theorem, the following corollary is immediate:

COROLARY. If for the long (short) simple roots α_i we have:

$$<\Omega,\alpha_i{}^v>\;\geq\;<\Lambda,\alpha_{hl(hs)}{}^v> \tag{4.6}$$

and if:

$$(\Lambda)\otimes(\Omega)=\bigoplus_{\Gamma\in\Lambda^+}m_\Gamma(\Gamma) \tag{4.7}$$

then :

$$(\Lambda)\otimes(\Omega+\Lambda_i)=\bigoplus_{\Gamma\in\Lambda^+}m_{\Gamma+\Lambda_i}(\Gamma+\Lambda_i) \tag{4.8}$$

In particular, if:

$$(\Lambda)\otimes(\Lambda_i)=\bigoplus_{\Gamma\in\Lambda^+}m_\Gamma(\Gamma) \tag{4.9}$$

then:

$$(\Lambda)\otimes(m\Lambda_i)=\underset{\Gamma\in\Lambda^+}{\oplus} m_{\Gamma+(m-1)\Lambda_i}(\Gamma+(m-1)\Lambda_i) \tag{4.10}$$

5. RESULTS

To obtain classical limits for relations of degrees three and four satisfied by the Hannabuss operator we proceed in the following way: We find out pairs (i,j) such that the Kronecker products of type $(\Lambda_i)\otimes(m\Lambda_j)$ $(m=1,2,...)$ possess the following properties: a) For all values of m the products associated with a pair (i,j) admit CG decompositions of equal lengths (three or four, respectively); b) Their CG decompositions are the same (modulo m) for any rank. Sets of Kronecker products with these properties are obtained using the above given generalization of a theorem of Feingold. Calculations have been done for algebras B_n, C_n and D_n for whivh we give below the Kronecker products and the classical limits in matrix form.

Lie algebras B_n

$$(\Lambda_1)\otimes(m\Lambda_k)=(\Lambda_1+m\Lambda_k)\oplus((m-1)\Lambda_k+\Lambda_{k+1})\oplus(\Lambda_{k-1}+(m-1)\Lambda_k);\ (k\le n-2)$$

Classical limit: $K_{\Lambda_1}{}^3-K_{\Lambda_1}=0$.

$$(\Lambda_1)\otimes(m\Lambda_{n-1})=(\Lambda_1+m\Lambda_{n-1})\oplus((m-1)\Lambda_{n-1}+2\Lambda_n)\oplus(\Lambda_{n-2}+(m-1)\Lambda_{n-1})$$

Classical limit: $K_{\Lambda_1}{}^3-K_{\Lambda_1}=0$.

$$(\Lambda_1)\otimes(m\Lambda_n)=(\Lambda_1+m\Lambda_n)\oplus(m\Lambda_n)\oplus(\Lambda_{n-1}+(m-2)\Lambda_n);\quad m\ge 2$$

Classical limit: $K_{\Lambda_1}{}^3-\frac{1}{4}K_{\Lambda_1}=0$.

$$(m\Lambda_2)\otimes(\Lambda_n)=(\Lambda_n+m\Lambda_2)\oplus(\Lambda_1+(m-1)\Lambda_2+\Lambda_n)\oplus(\Lambda_n+(m-1)\Lambda_2)$$

Classical limit: $K_{\Lambda_n}{}^3-K_{\Lambda_n}=0$.

$$\begin{aligned}&(m\Lambda_1)\otimes(\Lambda_k)=(m\Lambda_1+\Lambda_k)\oplus((m-1)\Lambda_1+\Lambda_{k+1})\oplus((m-1)\Lambda_1+\Lambda_{k-1})\oplus\\&((m-2)\Lambda_1+\Lambda_k);\quad (1<k\le n-2;\ m>2))\end{aligned}$$

Classical limit: $K_{\Lambda_k}{}^4-K_{\Lambda_k}{}^2=0$.

$$\begin{aligned}&(m\Lambda_1)\otimes(\Lambda_{n-1})=(m\Lambda_1+\Lambda_{n-1})\oplus((m-1)\Lambda_1+2\Lambda_n)\oplus(\Lambda_{n-2}+(m-1)\Lambda_1)\oplus\\&((m-2)\Lambda_1+\Lambda_{n-1});\quad (m\ge 2)\end{aligned}$$

Classical limit: $K_{\Lambda_{n-1}}{}^4-K_{\Lambda_{n-1}}{}^2=0$.

$$\begin{aligned}&(m\Lambda_1)\otimes(2\Lambda_n)=(m\Lambda_1+2\Lambda_n)\oplus((m-1)\Lambda_1+2\Lambda_n)\oplus(\Lambda_{n-1}+(m-1)\Lambda_1)\oplus\\&((m-2)\Lambda_1+2\Lambda_n);\quad (m\ge 2)\end{aligned}$$

Classical limit: $K_{2\Lambda_n}{}^4-K_{2\Lambda_n}{}^2=0$.

Lie algebras C_n

$$(\Lambda_1)\otimes(m\Lambda_k)=(\Lambda_1+m\Lambda_k)\oplus((m-1)\Lambda_k+\Lambda_{k+1})\oplus(\Lambda_{k-1}+(m-1)\Lambda_k);\ (1\le k\le n)$$

Classical limit: ${K_{\Lambda_1}}^3-K_{\Lambda_1}=0$.

$$(m\Lambda_1)\otimes(\Lambda_n)=(m\Lambda_1+2\Lambda_n)\oplus((m-1)\Lambda_1+\Lambda_n)\oplus(\Lambda_{n-1}+(m-1)\Lambda_1)\oplus ((m-2)\Lambda_1+\Lambda_n);\quad (m\ge 2)$$

Classical limit: ${K_{\Lambda_n}}^3-K_{\Lambda_n}=0$.

$$(\Lambda_2)\otimes(m\Lambda_n)=(m\Lambda_n+\Lambda_2)\oplus(\Lambda_1+(m-1)\Lambda_n+\Lambda_{n-1})\oplus(\Lambda_{n-2}+(m-1)\Lambda_n)$$

Classical limit: ${K_{\Lambda_2}}^3-4K_{\Lambda_2}=0$.

$$(m\Lambda_1)\otimes(\Lambda_k)=(m\Lambda_1+\Lambda_k)\oplus((m-1)\Lambda_1+\Lambda_{k+1})\oplus(\Lambda_{k-1}+(m-1)\Lambda_1)\oplus ((m-2)\Lambda_1+\Lambda_k);\ (1<k\le n-1; m\ge 2)$$

Classical limit: ${K_{\Lambda_k}}^4-{K_{\Lambda_k}}^2=0$.

$$(2\Lambda_1)\otimes(m\Lambda_n)=(2\Lambda_1+m\Lambda_n)\oplus(\Lambda_1+\Lambda_{n-1}+(m-1)\Lambda_n)\oplus(m\Lambda_n)\oplus (2\Lambda_1+(m-2)\Lambda_n);\quad (m\ge 2)$$

Classical limit: ${K_{2\Lambda_1}}^4-4{K_{2\Lambda_1}}^2=0$.

$$(\Lambda_3)\otimes(m\Lambda_n)=(\Lambda_3+m\Lambda_n)\oplus(\Lambda_2+\Lambda_{n-1}+(m-1)\Lambda_n)\oplus(\Lambda_1+\Lambda_{n-1}+(m-1)\Lambda_n)\oplus (\Lambda_{n-3}+(m-1)\Lambda_n)$$

Classical limit: ${K_{\Lambda_3}}^4-10{K_{\Lambda_3}}^2+9I=0$.

$$(\Lambda_1)\otimes(m\Lambda_k+\Lambda_n)=(\Lambda_1+m\Lambda_k+\Lambda_n)\oplus(\Lambda_{k+1}+\Lambda_n+(m-1)\Lambda_k)\oplus(m\Lambda_k+\Lambda_{n-1})\oplus (\Lambda_{k-1}+(m-1)\Lambda_k+\Lambda_n)$$

Classical limit: ${K_{\Lambda_1}}^4-{K_{\Lambda_1}}^2=0$.

Lie algebras D_n

$$(\Lambda_1)\otimes(m\Lambda_k)=(\Lambda_1+m\Lambda_k)\oplus((m-1)\Lambda_k+\Lambda_{k+1})\oplus(\Lambda_{k-1}+(m-1)\Lambda_k);\ (1\le k\le n-2)$$

Classical limit: ${K_{\Lambda_1}}^3-K_{\Lambda_1}=0$.

$$(\Lambda_1)\otimes(m\Lambda_{n-2})=(\Lambda_1+m\Lambda_{n-2})\oplus((m-1)\Lambda_{n-2}+\Lambda_{n-1}+\Lambda_n)\oplus(\Lambda_{n-3}+(m-1)\Lambda_{n-2})$$

Classical limit: ${K_{\Lambda_1}}^3-K_{\Lambda_1}=0$.

$$(m\Lambda_2)\otimes(\Lambda_n)=(\Lambda_n+m\Lambda_2)\oplus(\Lambda_1+(m-1)\Lambda_2+\Lambda_{n-1})\oplus(\Lambda_n+(m-1)\Lambda_2)$$

Classical limit: ${K_{\Lambda_n}}^3-K_{\Lambda_n}=0$.

$$(m\Lambda_1)\otimes(\Lambda_k)=(m\Lambda_1+\Lambda_k)\oplus((m-1)\Lambda_1+\Lambda_{k+1})\oplus(\Lambda_{k-1}+(m-1)\Lambda_1)\oplus$$
$$((m-2)\Lambda_1+\Lambda_k);\ (k<n-2;\ m>2)$$

Classical limit: $K_{\Lambda_k}{}^4-K_{\Lambda_k}{}^2=0$.

$$(m\Lambda_3)\otimes(\Lambda_n)=(m\Lambda_3+\Lambda_n)\oplus(\Lambda_2+\Lambda_{n-1}+(m-1)\Lambda_3)\oplus(\Lambda_1+\Lambda_n+(m-1)\Lambda_3)\oplus$$
$$(\Lambda_{n-1}+(m-1)\Lambda_3)$$

Classical limit: $K_{\Lambda_n}{}^4-\frac{5}{2}K_{\Lambda_n}{}^2+\frac{9}{16}I=0$.

The matrix identities $P(K_{\Lambda_i})=0$ obtained from the minimal polynomials satisfied by $O_{\Lambda_i,m\Lambda_j}$ are equations of co-adjoint orbits through the higest weight Λ_j associated with the representation Λ_i. For the case $i=1$ a simple explanation can be given for the form of these equations. Let us consider indeed the matrices K_{Λ_1}, which for the algebras C_n and D_n are of the form:

$$K_{\Lambda_1}=\begin{pmatrix} A & B \\ -C & -A^t \end{pmatrix}$$

where A, B and C are $n\ x\ n$ matrices and $B=B^t$, $C=C^t$ for C_n algebras and $B=-B^t$, $C=-C^t$ for D_n algebras. Thus, for the algebras C_n, the matrix K_{Λ_1} has $n(2n+1)$ distinct matrix elements anf for the algebras D_n, the matrix K_{Λ_1} has $n(2n-1)$ distinct matrix elements. These are precisely the dimensions of the corresponding adjoint representations of these algebras. Indeed, for C_n algebras we have $\dim(2\Lambda_1)=n(2n+1)$ and, for D_n $(n\geq 4)$, $\dim(\Lambda_2)=n(2n-1)$.
Let us now consider the matrix:

$$K_{\Lambda_1}{}^2=\begin{pmatrix} A^2-BC & AB-BA^t \\ -CA+A^tC & (A^2-BC)^t \end{pmatrix}$$

The submatrices $AB=-BA^t$ and $-CA+A^tC$ are antisymmetric for the algebras C_n and symmetric for the algebras D_n i.e. they have $n(2n-1)$ and $n(2n+1)$ distinct matrix elements for C_n and D_n algebras, respectively. These are precisely the dimensions of representations $(0)\oplus(\Lambda_2)=((\Lambda_1)\otimes(\Lambda_1))_{antisymm}$ for C_n and $(0)\oplus(2\Lambda_1)=((\Lambda_1)\otimes(\Lambda_1))_{symm}$ for D_n. This phenomenon is general: odd powers of K_{Λ_1} behaves like K_{Λ_1}, i.e. tensors of type $(2\Lambda_1)$ for C_n and of type (Λ_2) for D_n; even powers behaves like $K_{\Lambda_1}{}^2$, i.e. like tensors of type $(0)\oplus(\Lambda_2)$ for C_n and of type $(2\Lambda_1)$ for D_n.

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