Anexa nr. 6

ACADEMIA ROMÂNĂ
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HABILITATION THESIS - ABSTRACT

TITLE: The Analysis of Some Classes of Nonlinear PDEs

Habilitation Field: Mathematics

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Bucureşti, 2021

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This habilitation thesis is dedicated to the study of some classes of nonlinear Partial Differential Equations (PDEs). A Partial Differential Equation (PDE) is a relation that contains the partial derivatives of an unknown function depending on at least two independent variables. The study of PDEs goes back to the eighteenth century and it is the result of the analytical investigation of a large set of physical models (see, for example, the works by Leonhard Euler, Jean-Baptiste le Rond d'Alembert, Joseph-Louis Lagrange, Gaspard Monge, Pierre-Simon Laplace, Augustin-Louis Cauchy, Carl Gustav Jacob Jacobi, William Rowan Hamilton). Since the middle of the nineteenth century (see, for example, the works by Georg Friedrich Bernhard Riemann, Henri Poincaré, David Hilbert), PDEs have become an essential tool for studying problems in other branches of Mathematics. The main ideas on applications of PDEs were firstly indicated by Henri Poincaré in 1890 in his paper *Sur les Équations aux Dérivés Partielles de la Physique Mathématique* [20]. Although the origin of nonlinear PDEs is old, their investigation became an independent field that was expanded in many research directions during the last half of the twentieth century. Nonlinear PDEs arise in chemical and biological problems, in formulating fundamental laws of nature, in different areas of Physics, Applied Mathematics and Engineering (such as acoustics, fluid dynamics, solid mechanics, nonlinear optics, plasma physics, quantum field theory). The study of nonlinear PDEs is a very difficult task because there are no general methods to solve such equations. Each problem or each equation is unique because of its "nonlinearity" and new methods must be found permanently to solve at least a class of nonlinear PDEs. For more details on the history and development of PDEs we refer to the work by H. Brezis & F. Browder [3].

The main goal of this habilitation thesis is to present and analyse various classes of nonlinear PDEs, which reflect the main scientific contributions I obtained in the recent years after obtaining the Ph.D. degree (candidate's PhD Thesis [21] was defended in September 2012). I focus on theoretical aspects regarding some classes of PDEs with the auxiliary scope to underline important properties of their solutions such as well-posedness, existence or nonexistence, uniqueness, qualitative properties, asymptotic behavior of the sequence of solutions for some classes of PDEs.

This thesis is divided into four chapters.

Chapter 1 is entitled "*Nonlinear eigenvalue problems*" and in this chapter we study some nonlinear eigenvalue problems in bounded or unbounded domains from the Euclidean space $\mathbb{R}^N$. The differential operators involved in the construction of the eigenvalue problems analysed here are the Laplace operator,

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the p-Laplace operator, the fractional Laplace operator or the Baouendi-Grushin operator. The chapter contains 5 sections, each of them being based on a different article and describing a different problem.

Section 1.1 is based on paper [15]. Here, it is analysed a nontypical eigenvalue problem in a bounded domain from the Euclidean space $\mathbb{R}^2$ subject to the homogeneous Dirichlet boundary condition and it is showed that the spectrum of the problem contains two distinct intervals separated by an interval where there are no other eigenvalues.

Section 1.2 is based on paper [6]. In this section, an eigenvalue problem is studied in a smooth bounded domain subject to the homogeneous Neumann boundary condition. It is proved that this problem possesses a continuous family of eigenvalues plus one more isolated eigenvalue.

Section 1.3 is based on paper [14]. In this section, we perturbed with a p-Laplace operator the eigenvalue problem for the Laplace operator studied by Szulkin & Willem [24], namely

$$-\Delta u = \lambda V(x) u, \quad u \in \mathcal{D}_0^{1,2}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\nabla \cdot\|_{L^2(\Omega)}},$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is an open set and $V : \Omega \to [0, \infty)$ is a weight function which may have singular points satisfying the hypotheses

$$\begin{cases} V \in L^1_{loc}(\Omega),\ V = V_1 + V_2,\ V_1 \in L^{N/2}(\Omega), \\ \lim\limits_{|x|_N \to \infty} |x|_N^2 V_2(x) = 0,\ \lim\limits_{x \to y} |x-y|_N^2 V_2(x) = 0 \ \text{ for any } \ y \in \overline{\Omega}. \end{cases}$$

The new family of problems is $-\Delta u - \Delta_p u = \lambda V(x) u$ with $p \in (1, N) \setminus \{2\}$ and this family is studied in an Orlicz-Sobolev setting on general open sets from $\mathbb{R}^N$ with $N \geq 3$. The analysis of these problems leads to a full characterization of the spectrum as being an unbounded open interval.

Section 1.4 is based on paper [7] and is concerned with the existence of nontrivial solutions for a perturbation of the eigenvalue problem of the fractional (s,p)-Laplacian operator $(-\Delta_p)^s u = \lambda |u|^{p-2} u$, in Ω, $u = 0$, in $\mathbb{R}^N \backslash \Omega$, with a fractional (t,q)-Laplacian operator in the left-hand side of the equation, when $s, t \in (0,1)$, $p, q \in (1, \infty)$ are such that $s - N/p = t - N/q$ and $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with Lipschitz boundary. The nontrivial solutions of the perturbed eigenvalue problem exists if and only if parameter λ is strictly larger than the first eigenvalue of the fractional (s,p)-Laplacian.

Section 1.5 is based on paper [17]. It is shown that the spectrum of a nonhomogeneous Baouendi-Grushin type operator subject to the homogeneous Dirichlet boundary condition is exactly the open interval $(0, \infty)$. This is in sharp contrast with the situation when we deal with the "classical" Baouendi-Grushin operator when the spectrum is an increasing and unbounded sequence of positive real numbers. In addition, we showed that for each eigenvalue of Baouendi-Grushin nonhomogeneous operator there exists a sequence of eigenfunctions converging to zero.

Note that the results from Section 1.2 are also included in [4] while the the results from Section 1.4 are also included in [5].

Chapter 2 entitled "*The asymptotic behavior of solutions for some classes of PDEs*" contains 6 sections each of them discussing a different problem.

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Section 2.1 is based on paper [18] and is dedicated to the study of the equation $-\Delta_p u = \lambda e^u$ in Ω subject to the homogeneous Dirichlet boundary condition, where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary. It is proved that for each real number $p > N$, there exists a positive real number $\lambda^\star$, independent on p, such that for each $\lambda \in (0, \lambda^\star)$, this problem possesses a nonnegative solution u_p. Also, the asymptotic behavior of the sequence $\{u_p\}$ as $p \to \infty$ is analysed and it is shown that sequence $\{u_p\}$ converges uniformly to the distance function to the boundary of the domain.

Section 2.2 is based on paper [8] and is devoted to the study of the family of eigenvalue problems

$$\begin{cases} -\Delta_{2n} u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ ||u||_{L^2(\Omega)} = 1, & \end{cases} \tag{1}$$

where Δ_{2n} stands for the $2n$-Laplace operator and $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary. Regarding the above family of problems (1), it is proved that the corresponding sequence of positive eigenfunctions associated to the lowest eigenvalue converges uniformly in our domain, as $n \to \infty$, to the normalized distance function to the boundary in $L^2(\Omega)$.

Section 2.3 is based on paper [13]. Here, the existence of principal eigenvalues and eigenfunctions for a family of eigenvalue problems described by a system consisting in two partial differential equations involving p-Laplacians is investigated. Next, the asymptotic behaviour, as $p \to \infty$, of the sequence of principal eigenfunctions is studied and it is proved that, passing eventually to a subsequence, it converges uniformly to a certain limit given by a pair of continuous functions. Moreover, the limiting equations which have as solutions the limiting functions are identified.

Section 2.4 is based on paper [11]. In this section, we analyse the asymptotic behavior of the sequence $\{v_n\}$ of nonnegative solutions for a class of inhomogeneous problems settled in some Orlicz-Sobolev spaces subject to the homogeneous Dirichlet boundary condition and we show that $\{v_n\}$ converges uniformly in domain Ω, as $n \to \infty$, to the distance function to the boundary of the domain.

Section 2.5 is based on paper [2]. The asymptotic behavior of the sequence $\{u_n\}$ of positive first eigenfunctions for a class of inhomogeneous eigenvalue problems is studied in the setting of Orlicz-Sobolev spaces. After possibly extracting a subsequence, we prove that $u_n \to u_\infty$ uniformly in Ω as $n \to \infty$, where u_∞ is a nontrivial viscosity solution of a nonlinear PDE involving the ∞-Laplacian, which is identified.

Section 2.6 is based on paper [22] and here, it is shown that the sequence $\{u_n\}$ of solutions for a class of inhomogeneous problems converges uniformly in domain Ω as $n \to \infty$ to an ∞-harmonic function satisfying the prescribed Dirichlet data on the boundary.

Note that the results from Section 2.2 are also included in [5] while the the results from Section 2.4 are also included in [10].

Chapter 3 is entitled "*Torsional Creep type Problems*" and is devoted to the analysis of some problems which are related to torsional creep phenomenon, which represents the permanent plastic

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deformation of a material subject to a torsional moment for an extended period of time and at sufficiently high temperature.

Section 3.1 is based on paper [23]. In this section, we study the asymptotic behaviour of the sequence of positive solutions for a family of torsional creep-type problems involving anisotropic rapidly growing differential operators. We show that the sequence of solutions converges uniformly on the domain to a certain distance function to the boundary defined in accordance with the anisotropy of the analysed problem.

Section 3.2 is based on paper [16]. Here, we investigate the asymptotic behaviour of solutions for a family of torsional creep problems involving the Grushin p-Laplacian. These results complement some earlier works on the topic by L. E. Payne & G. A. Philippin [19], B. Kawohl [12] and T. Bhattacharya, E. DiBenedetto, & J. Manfredi [1].

Section 3.3 is based on paper [9]. This section is concerned with the asymptotic behavior of solutions to a family of boundary value problems involving differential operators in divergence form on a domain equipped with a Finsler metric. Solutions are shown to converge uniformly to the distance function to the boundary of the domain which takes into account the Finsler norm involved in the equation. This implies that a well-known result in the analysis of problems modelling torsional creep phenomena continues to hold in this more general setting.

Note that the results from Section 3.3 are also included in [5].

The **last chapter** of the thesis is entitled "*Final comments and further directions of research*" and presents some ideas regarding the study of some open problems starting from the research presented in the first three chapters of this thesis. Further plans regarding the evolution of the professional and scientific career of the candidate will also be presented both from the point of view of research and teaching.

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Date: 27.09.2021

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