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Quasi-Log Varieties

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We extend the Cone and Contraction Theorems of the Log Minimal Model Program to log varieties with arbitrary singularities.

INTRODUCTION

The starting point of the Minimal Model Program is the Cone and Contraction Theorems of S. Mori: the K_X-negative part of the cone of effective curves of a nonsingular projective 3-fold X is locally rationally polyhedral, with contractible faces. One hopes that, by replacing the original variety with the target space of the contraction associated to a negative face, or a small modification of it (a flip), one reaches a minimal model or a Mori–Fano fiber space after finitely many steps. These intermediate varieties have singularities in dimension at least three; thus, it is clear that one must consider varieties with some mild singularities in order to find minimal models.

In characteristic zero, Y. Kawamata, X. Benveniste, M. Reid, V.V. Shokurov, and J. Kollár proved the Cone and Contraction Theorems for varieties with Kawamata log terminal singularities. This part of the Log Minimal Model Program was expected to work for log varieties with arbitrary singularities, under certain assumptions on rays or their contractions. This is our main result, and, before we state it, we make the following definition:

Definition 1. A *generalized log variety* (X,B) is a pair consisting of a normal variety X and an effective Weil $\mathbb{R}$-divisor B such that $K+B$ is $\mathbb{R}$-Cartier. We denote by $(X,B)_{-\infty}$ the locus where (X,B) does not have log canonical singularities (it has a natural subscheme structure). A *log variety* is a generalized log variety that has log canonical singularities, i.e., $(X,B)_{-\infty}=\varnothing$.

Theorem 2. *Let (X,B) be a projective generalized log variety defined over a field of characteristic zero. Let $\overline{\mathrm{NE}}(X)$ be the closure of the cone of effective curves of X, and set*

$$\overline{\mathrm{NE}}(X)_{-\infty}=\mathrm{Im}(\overline{\mathrm{NE}}((X,B)_{-\infty})\to\overline{\mathrm{NE}}(X)).$$

(i) *Let F be a face of the cone $\overline{\mathrm{NE}}(X)$ such that*

$$F\cap(\overline{\mathrm{NE}}(X)_{-\infty}+\overline{\mathrm{NE}}(X)_{K+B\geq 0})=\{0\}.$$

Then there exists a projective contraction $\varphi_F\colon X\to Y$ that contracts exactly the curves belonging to F. Furthermore, φ_F restricted to $(X,B)_{-\infty}$ is a closed embedding.

(ii) $\overline{\mathrm{NE}}(X)=\overline{\mathrm{NE}}(X)_{K+B\geq 0}+\overline{\mathrm{NE}}(X)_{-\infty}+\sum R_j$, *where the R_j's are the one-dimensional faces satisfying the assumption in* (i). *Furthermore, the R_j's are discrete in the half-space $N_1(X)_{K+B<0}$.*

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This result is a special case of Theorem 5.10. As a corollary, we generalize a result of J. Kollár [12] (in characteristic zero): if (X, B) has log canonical singularities outside a finite set of points, the Cone Theorem holds exactly as in the Kawamata log terminal case. In particular, this holds for a normal surface with $\mathbb{Q}$-Gorenstein singularities (cf. [16]). See also [18] for applications.

We also establish the Base Point Free Theorem for generalized log varieties, including the log big case (Theorems 5.1 and 7.2). Another application is the uniqueness of minimal lc centers of (quasi-)log Fano varieties (Theorem 6.6).

For the proof, it turns out to be easier to work in a larger class of varieties that we call *quasi-log varieties*. Their definition is motivated by Y. Kawamata's X-method, which produces global sections of adjoint line bundles L: we first create singularities that are not Kawamata log terminal inside X, i.e., $\mathrm{LCS}(X) \neq \varnothing$. By adjunction, we expect that $L|_{\mathrm{LCS}(X)}$ is still an adjoint line bundle; hence, if it has a global section (by induction, for instance), we can lift it to a global section of L by the Kawamata–Viehweg vanishing. Unlike the given variety, its LCS locus is no longer normal, not even irreducible or equi-dimensional, and its log canonical class in the usual sense does not make sense either. However, by definition, the LCS locus is the target space of a 0-log contraction (cf. [18, 3.27(ii)]) from a variety with only embedded normal crossing singularities. We call *quasi-log varieties* those varieties that appear as the target space of such contractions. Examples are varieties with embedded normal crossing singularities, generalized log varieties, and their LCS loci (see Examples 4.3).

A quasi-log variety X is endowed with an $\mathbb{R}$-Cartier divisor ω, the descent of the log canonical class of the total space of the 0-log contraction, a closed proper subscheme $X_{-\infty} \subset X$, and a finite family $\{C\}$ of reduced and irreducible subvarieties of X. We say that ω is the *quasi-log canonical class* of X, $X_{-\infty}$ is the locus where X does not have *qlog canonical* singularities, and the C's are the *qlc centers* of X. The open subset $X \setminus X_{-\infty}$ is reduced, with seminormal singularities. We note here that singularities appearing on special LCS loci have been called *semi-log canonical* in the literature.

The adjunction and vanishing for quasi-log varieties are proved in Theorem 4.4. The former holds by the very definition, while the latter is an extension to normal crossing pairs of the vanishing and torsion freeness theorems of J. Kollár, based on previous work by Y. Kawamata, H. Esnault, and E. Viehweg. Applied to log varieties, our vanishing theorem is stronger than the Kawamata–Viehweg (or Nadel) vanishing.

We expect that normal quasi-log varieties are *equivalent* (cf. Example 4.3.1) to generalized log varieties according to the Adjunction Conjecture. We only have partial results in this direction (cf. Proposition 4.7, Theorem 4.9, and Remarks 4.10). One should also note that, if the Adjunction Conjecture holds, the X-method works inductively in the category of log varieties, as long as we restrict to normal lc centers.

Finally, for technical reasons, we require that our varieties with normal crossing singularities are globally embedded as hypersurfaces. This is enough for applications to generalized log varieties; however, we expect that this extra assumption is not necessary (see Remark 2.9).

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1. PRELIMINARY

A *variety* is a scheme of finite type defined over an algebraically closed field k of characteristic zero. We denote by $\mathrm{Div}(X)$ the abelian group of Cartier divisors of X. A K-*Cartier divisor* on X is an element of $\mathrm{Div}(X)_K := \mathrm{Div}(X) \otimes_{\mathbb{Z}} K$, for $K \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$.

Let $\pi\colon X \to S$ be a proper morphism of varieties. We denote by $Z_1(X/S)$ the abelian group generated by proper integral curves in X mapped to points by π. The natural pairing $\mathrm{Div}(X) \times Z_1(X/S) \to \mathbb{Z}$ induces, via numerical equivalence and tensoring with $\mathbb{R}$, a perfect pairing of finite-dimensional $\mathbb{R}$-vector spaces $N^1(X/S) \times N_1(X/S) \to \mathbb{R}$. We denote by $\mathrm{NE}(X/S) \subset N_1(X/S)$ the cone generated by proper integral curves in X mapped to points by π, and by $\overline{\mathrm{NE}}(X/S)$ its closure in the real topology. The dual of $\mathrm{NE}(X/S)$ in $N^1(X/S)$ is called the *relatively nef cone*. The *relatively ample cone* $\mathrm{Amp}(X/S)$ is the cone of $N^1(X/S)$ generated by classes of relatively ample Cartier divisors (if any). A K-Cartier divisor D is *relatively nef* (*ample*) if its class in $N^1(X/S)$ belongs to the relatively nef (ample) cone. If X/S is projective, S. Kleiman proved that the relatively ample cone is the interior of the relatively nef cone. In particular, a K-Cartier divisor D is relatively ample if and only if $(D \cdot z) > 0$ for all $z \in \overline{\mathrm{NE}}(X/S) \setminus \{0\}$.

An $\mathbb{R}$-Cartier divisor D is *relatively semiample* if $D \sim_{\mathbb{R}} f^*H$, where $f\colon X/S \to Y/S$ is a proper morphism and H is a relatively ample $\mathbb{R}$-Cartier divisor. If $D \in \mathrm{Div}(X)_{\mathbb{Q}}$, this is equivalent to the surjectivity of the natural map $\pi^*\pi_*\mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$ for some large and divisible positive integer m.

An open subset $U \subseteq X$ is called *big* if $X \setminus U$ has codimension at least two in X.

2. NORMAL CROSSING PAIRS

Definition 2.1. A variety X has *multicrossing* singularities if, for every closed point $x \in X$, there exist integers N, l, subsets $I_1, \ldots, I_l$ of $\{0, \ldots, N\}$, and an isomorphism of complete local rings

$$\mathcal{O}^{\wedge}_{X,x} \xrightarrow{\sim} \frac{k[[x_0, \ldots, x_N]]}{(\prod_{i \in I_1} x_i, \ldots, \prod_{i \in I_l} x_i)}.$$

If $l = 1$ for every $x \in X$, we say that X has *normal crossing* singularities. Furthermore, if each irreducible component of X is nonsingular, we say that X is a *simple multicrossing* (*normal crossing*) variety.

For a scheme X, we denote by $\epsilon\colon X_\bullet \to X$ the associated simplicial scheme $((X_0/X)^{\Delta_n} \to X)_{n \geq 0}$. Here $\epsilon = \{\epsilon_n\}$, where $\epsilon_0\colon X_0 \to X$ is the normalization and ϵ_n is the natural projection. The simplicial maps are $\delta_i\colon X_{n+1} \to X_n$, $x_0 \times \ldots \times x_{n+1} \mapsto x_0 \times \ldots \widehat{x}_i \ldots \times x_n$ and $s_i\colon X_n \to X_{n+1}$, $x_0 \times \ldots \times x_n \mapsto x_0 \times \ldots \times x_i \times x_i \times x_{i+1} \times \ldots \times x_n$. This is a proper hypercovering [3, 7]. A *stratum* of X is, by definition, the image on X of some irreducible component of $X_\bullet$.

Lemma 2.2. *The following hold for a variety X with multicrossing singularities*:

(i) *The associated hypercovering $\epsilon\colon X_\bullet \to X$ is proper, smooth, and of cohomological descent with respect to locally free sheaves on X.*

(ii) *We have an isomorphism of functors* $\mathrm{Hom}(X, \cdot) \xrightarrow{\sim} \mathrm{Hom}(X_\bullet, \cdot)$.

(iii) *X has seminormal singularities.*

(iv) *If X is a simple multicrossing variety, each stratum of X is nonsingular.*

Proof. (i) Each ϵ_n is a finite map and, hence, is proper. It is also easy to see that each X_n is nonsingular: in the notations of Definition 2.1, for $\alpha \in I_1 \times \ldots \times I_l$, denote $\{\alpha\} = \{\alpha_1, \ldots, \alpha_l\} \subset \Delta_N$. Also, denote by J the set of all elements of $I_1 \times \ldots \times I_l$ that are minimal with respect to the partial order $\alpha \le \beta$ if and only if $\{\alpha\} \subseteq \{\beta\}$. Then, at the complete local rings level, X_n is the spectrum of

$$\sum_{q\colon \Delta_n \to J} \frac{k[[x_0, \dots, x_N]]}{(x_i \colon i \in \{q(0)\} \cup \ldots \cup \{q(n)\})}.$$

Finally, cohomological descent for a locally free sheaf F on X means that $F \xrightarrow{\sim} R^{\bullet}\epsilon_*(\epsilon^* F)$. Since ϵ is finite, it is enough to show that the natural map $\mathcal{O}_X \to \epsilon_* \mathcal{O}_{X_\bullet}$ is an isomorphism. This is a local statement, and it can be checked as in [7, 4.1].

(ii) A morphism $f\colon X \to Y$ induces $f\colon X_\bullet \to Y$ with components $f_n = f \circ \epsilon_n$. Conversely, let $f\colon X_\bullet \to Y$ be a morphism. The induced map $f\colon X \to Y$ is defined set-theoretically by $f(x) := f_0(\epsilon_0^{-1}(x))$. This map is well defined since any two points in the fiber of ϵ_0 are the images of some point on some X_n under different compositions of δ_i's. Moreover, f is a morphism since, for every $h \in \mathcal{O}_Y$, $f_0^*(h) \in \mathcal{O}_{X_0}$ takes the same value on the glueing data, thus, it belongs to $\mathcal{O}_X \subset \mathcal{O}_{X_0}$.

(iii) See [2].

(iv) The normalization ϵ_0 is a disjoint union of embeddings. Therefore, the same holds for ϵ_n, $n \ge 1$. Each X_n is smooth since X has multicrossing singularities; hence, all strata are smooth. In this case, the strata are the components of the intersections of irreducible components of X. $\square$

Let X be a variety with multicrossing singularities. A Cartier divisor D on X is called *permissible* if it induces a Cartier divisor $D^\bullet$ on $X_\bullet$, i.e., $D^n = \epsilon_n^* D$ is a Cartier divisor on X_n for every n (equivalently, D contains no strata of X in its support). We say that D is a *multicrossing divisor* on X if, in the notations of Definition 2.1, we have

$$\mathcal{O}^\wedge_{D,x} \xrightarrow{\sim} \frac{k[[x_0, \dots, x_N]]}{(\prod_{i\in I_1} x_i, \dots, \prod_{i \in I_l} x_i, \prod_{i\in I'} x_i)},$$

where $I' \subset \Delta_N$ and $I' \cap \bigcup_{j=1}^l I_j = \varnothing$. We denote by $\mathrm{Div}_0(X)$ the free abelian group generated by all permissible Cartier divisors on X. A permissible K-divisor on X is an element of $\mathrm{Div}_0(X) \otimes_{\mathbb{Z}} K$, for $K \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$. For a permissible K-divisor $D = \sum_i d_i D_i$, its *reduced part* is $\sum_{d_i = 1} D_i$. We denote $D^{>1} = \sum_{d_i > 1} d_i D_i$ and $D^{<1} = \sum_{d_i<1} d_i D_i$. We say that D is a *boundary* (*subboundary*) if $0 \le d_i \le 1$ $\forall i$ ($d_i \le 1$ $\forall i$).

Definition 2.3. A *multicrossing pair* (X, B) is a multicrossing variety X endowed with a permissible $\mathbb{R}$-divisor B whose support is a multicrossing divisor on X. If X has normal crossing singularities, we say that (X, B) is a *normal crossing pair*.

A *stratum of* (X, B) is a stratum either of X or of the reduced part of B. Equivalently, the strata are the images of strata of the log nonsingular pairs $\{(X_n, B^n)\}_{n \ge 0}$. For instance, the maximal strata of (X, B) are the irreducible components of X.

Remark 2.4. Compared with the *generalized normal crossing varieties* introduced by Y. Kawamata [7], the ambient space X of a normal crossing pair has generalized normal crossing singularities; however, B has arbitrary coefficients in our case.

Lemma 2.5. *The following properties hold for a multicrossing pair* (X, B):

(i) *Each stratum is irreducible, with multicrossing singularities. A stratum that is minimal* (*with respect to inclusion*) *is nonsingular.*

(ii) *There are only finitely many strata.*

(iii) *The nonempty intersection of any two strata is a union of strata. In particular, minimal strata are mutually disjoint.*

We say that a permissible divisor D has *multicrossing support on* (X,B) if it contains no strata of (X,B) and both D and its restriction to the reduced part of B have multicrossing support. A variety with normal crossings X is a locally complete intersection; hence, it has an invertible dualizing sheaf $\mathcal{O}_X(K)$. The *canonical divisor* $K \in \operatorname{Div}(X)$ is well defined up to linear equivalence.

Remark 2.6 (dévissage)**.** Let (X,B) be a normal crossing pair, and let Y be a union of irreducible components of X. Denote by X' the union of the other irreducible components of X and write $B_Y = B|_Y + X'|_Y$, $B_{X'} = Y|_{X'} + B|_{X'}$. Then the following hold:

(i) (Y, B_Y) and $(X', B_{X'})$ are normal crossing pairs.

(ii) $(K+B)|_Y = K_Y + B_Y$ and $(K+B)|_{X'} = K_{X'} + B_{X'}$.

(iii) $\mathcal{I}_{Y,X} \simeq j_*\mathcal{O}_{X'}(-Y|_{X'})$, where $j\colon X' \to X$ is the inclusion.

In particular, let L be a Cartier divisor on X such that $L = K + B + H$. Denote $L' = L|_{X'} - Y|_{X'}$, so that $L' = K_{X'} + B|_{X'} + H|_{X'}$. Then we have a short exact sequence $0 \to j_*\mathcal{O}_{X'}(L') \to \mathcal{O}_X(L) \to \mathcal{O}_Y(L|_Y) \to 0$.

Definition 2.7. We say that a normal crossing pair (X,B) is *embedded* if there exists a closed embedding $j\colon X \to M$, where M is a nonsingular variety of dimension $\dim X + 1$.

Let (X,B) be an embedded normal crossing pair, and let C be a nonsingular stratum. The *embedded log transformation of* (X,B) *in* C, denoted $\sigma\colon (Y, B_Y) \to (X,B)$, is defined as follows: let $X \subset M$ be an embedding of X as a hypersurface in a nonsingular ambient space M. We denote by Y the reduced structure of the total transform of X in the blow-up of M in C. The morphism $\sigma\colon Y \to X$ is projective, Y has normal crossing singularities, the formula $\sigma^*(K+B) = K_Y + B_Y$ defines a divisor B_Y on Y, and the following properties hold:

(i) (Y, B_Y) is an embedded normal crossing pair.

(ii) The strata of (X,B) are exactly the images of the strata of (Y,B_Y).

(iii) $\mathcal{O}_X \xrightarrow{\sim} R^\bullet\sigma_*\mathcal{O}_Y$.

(iv) $\sigma^{-1}(C)$ is a maximal stratum of (Y, B_Y).

Proposition 2.8. *Let $X' \subset X$ be the union of some strata of an embedded normal crossing pair (X,B). Then there exist an embedded normal crossing pair (Y,B_Y) and a projective morphism $f\colon Y \to X$ such that*

(i) $\mathcal{O}_X \xrightarrow{\sim} R^\bullet f_*\mathcal{O}_Y$;

(ii) $f^*(K+B) = K_Y + B_Y$;

(iii) *the strata of (X,B) are exactly the images of the strata of (Y,B_Y);*

(iv) *$f^{-1}(X')$ is a union of maximal strata of (Y,B_Y).*

Proof. First, we may assume that each stratum of (X,B) is nonsingular. Indeed, after a finite number of embedded log transformations of X in its minimal strata, each irreducible component of X is nonsingular in the minimal strata of X, i.e., X has simple normal crossings. Similarly, the reduced part of B becomes simple multicrossings after a finite sequence of embedded log transformations of (X,B) in the minimal strata of B.

Once each stratum of (X,B) is nonsingular, we reach the conclusion after a finite number of embedded log transformations of (X,B) in the irreducible components of X'. $\square$

Remark 2.9. The embedded hypothesis is used to prove Proposition 2.8 and to resolve singularities of permissible subvarieties of a variety with normal crossings. Once the latter has been established, we expect our results to work for abstract normal crossing pairs.

3. VANISHING THEOREMS

We extend the vanishing and torsion freeness theorems of J. Kollár [11] to normal crossing pairs. The proof is based on logarithmic de Rham complexes, and we follow closely the presentation of [4]. See also [7].

Theorem 3.1. *Assume that (X,B) is an embedded normal crossing pair such that X is a proper variety and B is a boundary. Let L be a Cartier divisor on X, and let D be an effective Cartier divisor, permissible with respect to (X,B), with the following properties*:

(i) $L \sim_{\mathbb{R}} K + B + H$.

(ii) $H \in \mathrm{Div}(X)_{\mathbb{R}}$ *is semiample.*

(iii) $tH \sim_{\mathbb{R}} D + D'$ *for some positive real number t and for some effective $\mathbb{R}$-Cartier divisor D' permissible with respect to (X,B).*

Then the natural maps $H^q(X,\mathcal{O}_X(L)) \to H^q(X,\mathcal{O}_X(L+D))$ are injective for all q.

Proof. Blowing up X and incorporating the negative part of B into the pullback of L, we may assume that both (X,B) and $D+D'$ have normal crossing supports. Furthermore, we may assume that $H = aD + a'D'$, where $a > 0$, $a' \geq 0$, and $B' = B + aD + a'D'$ is a boundary with $\lfloor B' \rfloor = \lfloor B \rfloor$.

We have $L \sim_{\mathbb{R}} K + B'$. Since L and K are integral divisors, the set of boundaries having the same support and reduced part as B' and satisfying the above equality forms a rational polyhedron. After a perturbation of its fractional part, we may assume that B' is rational. In particular, $T = -L + K + B'$ is a $\mathbb{Q}$-Cartier divisor and $\nu T \sim 0$ for some positive integer ν. Assume that ν is minimal with this property. Denote $\mathcal{E} = \mathcal{O}_X(-L+K)$, and let R be the support of B'.

Let $X_\bullet \to X$ be the associated smooth, proper hypercovering. By Serre duality and cohomological descent, we have to check the surjectivity of the maps

$$H^q(X_\bullet, \mathcal{E}^\bullet(-D^\bullet)) \to H^q(X_\bullet, \mathcal{E}^\bullet).$$

We use the following commutative diagram:

$$\begin{array}{ccc} H^q(X_\bullet, \mathcal{E}^\bullet(-D^\bullet)) & \longrightarrow & H^q(X_\bullet, \mathcal{E}^\bullet) \\ \uparrow & & \uparrow \beta \\ \mathbf{H}^q(X_\bullet, \Omega^\bullet_{X_\bullet}(\log R^\bullet) \otimes \mathcal{E}^\bullet(-D^\bullet)) & \xrightarrow{\alpha} & \mathbf{H}^q(X_\bullet, \Omega^\bullet_{X_\bullet}(\log R^\bullet) \otimes \mathcal{E}^\bullet) \end{array}$$

Since $-L+K = \lfloor T \rfloor - \lfloor B \rfloor$, the restriction of $\mathcal{E}^\bullet$ to each component of $X_\bullet$ admits a logarithmic connection with poles along $R^\bullet$ whose residues along the components of $D^\bullet$ belong to the interval $(0,1)$ [4, 3.2]. By [4, 4.3], the map

$$\Omega^\bullet_{X_\bullet}(\log R^\bullet) \otimes \mathcal{E}^\bullet(-D^\bullet) \to \Omega^\bullet_{X_\bullet}(\log R^\bullet) \otimes \mathcal{E}^\bullet$$

is a quasi-isomorphism componentwise; thus, it is a quasi-isomorphism of simplicial complexes. Therefore, α is an isomorphism.

Let $\pi\colon Y_\bullet \to X_\bullet$ be the cyclic cover of degree ν corresponding to the torsion divisor $T^\bullet$. By [3], the spectral sequence

$$E_1^{pq} = H^q(Y_\bullet, \Omega^p_{Y_\bullet}(\log R^\bullet)) \Longrightarrow \mathbf{H}^{p+q}(Y_\bullet, \Omega^\bullet_{Y_\bullet}(\log R^\bullet))$$

degenerates. Since $\mathcal{E}^\bullet$ is a direct summand of $\pi_*\Omega^\bullet_{Y_\bullet}(\log R^\bullet)$, the spectral sequence

$$E_1^{pq} = H^q(X_\bullet, \Omega^p_{X_\bullet}(\log R^\bullet) \otimes \mathcal{E}^\bullet) \Longrightarrow \mathbf{H}^{p+q}(X_\bullet, \Omega^\bullet_{X_\bullet}(\log R^\bullet) \otimes \mathcal{E}^\bullet)$$

degenerates as well. Therefore, β is surjective. □

Theorem 3.2. *Let (Y,B) be an embedded normal crossing pair, and assume that B is a boundary. Let $f\colon Y\to X$ be a proper morphism, and let L be a Cartier divisor on Y such that $H=L-(K+B)$ is f-semiample.*

(i) *Every nonzero local section of $R^qf_*\mathcal{O}_Y(L)$ contains in its support the f-image of some strata of (Y,B).*

(ii) *Let $\pi\colon X\to S$ be a projective morphism, and assume that $H\sim_{\mathbb{R}} f^*H'$ for some π-ample $\mathbb{R}$-Cartier divisor H' on X. Then $R^qf_*\mathcal{O}_Y(L)$ is π_*-acyclic.*

Proof. (i) The conclusion is local; therefore, we may shrink X to an affine open subset and compactify it afterwards, so that X is projective, Y is proper, and H is semiample. If $R^qf_*\mathcal{O}_Y(L)$ admits a local section whose support does not contain any image of the (Y,B)-strata, one can find a very ample divisor A such that

- $H^0(X,R^qf_*\mathcal{O}_Y(L))\to H^0(X,R^qf_*\mathcal{O}_Y(L)\otimes\mathcal{O}_X(A))$ is not injective;
- f^*A is a permissible multicrossing divisor on (Y,B);
- the Leray spectral sequence of $L+f^*A$ with respect to f degenerates.

Replacing L by $L+f^*A$ if necessary, we may also assume that $H-f^*A$ is semiample. The degeneration of the Leray spectral sequence implies that the map $H^q(Y,\mathcal{O}_Y(L))\to H^q(Y,\mathcal{O}_Y(L+f^*A))$ is not injective, which contradicts Theorem 3.1.

(ii) Assume that $\dim S=0$, and let $H=f^*H_X$. If X has positive dimension, one can find a divisor A in some large, divisible multiple of H, such that its pullback $A'=f^*A$ is a permissible multicrossing divisor on (Y,B) and $R^qf_*\mathcal{O}_Y(L+A')$ is π_*-acyclic for all q. By (i), we have short exact sequences

$$0\to R^qf_*\mathcal{O}_Y(L)\to R^qf_*\mathcal{O}_Y(L+A')\to R^qf_*\mathcal{O}_{A'}(L+A')\to 0,$$

where $R^qf_*\mathcal{O}_Y(L+A')$ is π_*-acyclic by assumption, while $R^qf_*\mathcal{O}_{A'}(L+A')$ is π_*-acyclic by induction on X. Therefore, $E_2^{p,q}=0$ for $p\ge 2$ in the following commutative diagram of spectral sequences:

$$\begin{array}{ccc}
E_2^{p,q}=R^p\pi_*R^qf_*\mathcal{O}_Y(L) & \Longrightarrow & R^{p+q}(\pi\circ f)_*\mathcal{O}_Y(L)\\
\downarrow{\scriptstyle\varphi^{p,q}} & & \downarrow{\scriptstyle\varphi^{p+q}}\\
\overline{E}_2^{p,q}=R^p\pi_*R^qf_*\mathcal{O}_Y(L+A') & \Longrightarrow & R^{p+q}(\pi\circ f)_*\mathcal{O}_Y(L+A')
\end{array}$$

Since $E_2^{1,q}\to R^{1+q}(\pi\circ f)_*\mathcal{O}_Y(L)$ is injective, φ^{1+q} is injective by Theorem 3.1, and $\overline{E}^{1,q}=0$ by assumption, we obtain $E_2^{1,q}=0$.

Assume now that S is affine of positive dimension and $\pi\circ f$ surjects Y onto S. We use induction on the dimension of S.

a) Assume that each stratum of (Y,B) dominates a generic point of S. From the case $\dim S=0$, $R^p\pi_*R^qf_*\mathcal{O}_Y(L)$ $(p>0)$ does not contain any generic point of S in its support. Therefore, there exists a general hyperplane section A of S, containing the support of all these sheaves, such that its pullback A' on Y is a multicrossing divisor on (Y,B). The argument in (i) shows that $R^qf_*\mathcal{O}_Y(L)$ is π_*-acyclic, except that φ^{p+q} is injective by (i) now, and $R^p\pi_*R^qf_*\mathcal{O}_Y(L)\otimes\mathcal{O}_S(A)$ is zero by the choice of A.

b) Let Y' be the union of all strata of (Y,B) that are not mapped onto generic points of S. After a sequence of embedded log transformations, we may assume that Y' is a union of irreducible components of Y. By (i), we have exact sequences

$$0\to R^qf_*(\mathcal{I}_{Y'}(L))\to R^qf_*\mathcal{O}_Y(L)\to R^qf_*\mathcal{O}_{Y'}(L)\to 0.$$

From Remark 2.6, $R^qf_*(\mathcal{I}_{Y'}(L))\xrightarrow{\sim}R^qf_*\mathcal{O}_{Y''}(L'')$, where $L''=K_{Y''}+B|_{Y''}+f^*H$. The pair $(Y'',B|_{Y''})$ satisfies the hypothesis in a); hence, the first term is π_*-acyclic. The third is π_*-acyclic by induction; thus, $R^qf_*\mathcal{O}_Y(L)$ is π_*-acyclic. $\square$

4. QUASI-LOG VARIETIES

Definition 4.1. A *quasi-log variety* is a scheme X endowed with an $\mathbb{R}$-Cartier divisor ω, a proper closed subscheme $X_{-\infty} \subset X$, and a finite collection $\{C\}$ of reduced and irreducible subvarieties of X such that there exists a proper morphism $f\colon (Y, B_Y) \to X$ from an embedded normal crossing pair satisfying the following properties:

(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.

(2) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil)$ induces an isomorphism

$$\mathcal{I}_{X_{-\infty}} \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor).$$

(3) The collection of subvarieties $\{C\}$ coincides with the images of (X, B)-strata that are not included in $X_{-\infty}$.

We use the following terminology: the subvarieties C are the *qlc centers* of X, $X_{-\infty}$ is the *non-qlog canonical locus* of X, and $f\colon (Y, B) \to X$ is a *quasi-log resolution* of X. We say that X has *qlog canonical singularities* if $X_{-\infty} = \varnothing$. Note that a quasi-log variety X is the union of its qlc centers and $X_{-\infty}$. A *relative quasi-log variety* X/S is a quasi-log variety X endowed with a proper morphism $\pi\colon X \to S$.

For simplicity, we will refer to a quasi-log variety as X or (X, ω).

Remarks 4.2. (i) X has qlog canonical singularities if and only if B is a subboundary. Indeed, the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & f_*\mathcal{I}_N & \longrightarrow & f_*\mathcal{O}_Y & \longrightarrow & f_*\mathcal{O}_N & & \\
 & & \uparrow\simeq & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{I}_{X_{-\infty}} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X_{-\infty}} & \longrightarrow & 0
\end{array}$$

implies that $X_{-\infty} \cap f(Y) = f(N)$, where $N = \lfloor B_Y^{>1} \rfloor$. Note that $X_{-\infty} \subsetneq X$ by assumption, but $X_{-\infty}$ may contain irreducible components of X. Also, f may not be surjective (cf. Example 4.3.4).

(ii) If B is a subboundary, property (2) of Definition 4.1 says that the natural morphism $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil)$ is an isomorphism. In particular, f is a surjective map with connected fibers. Furthermore, X is seminormal by [2]. In general, the same holds over the open subset of qlog canonical singularities $U = X \setminus X_{-\infty}$.

(iii) The *quasi-log canonical class* ω is defined up to $\mathbb{R}$-linear equivalence. This is more general than the case of generalized log varieties, where the log canonical class $K + B$ is defined up to linear equivalence.

(iv) The quasi-log resolution plays a role similar to that of a log resolution. Embedded log transformations of (Y, B_Y), or blow-ups of Y in centers that contain no (Y, B_Y)-strata, leave the quasi-log structure on X invariant. Furthermore, we may slightly perturb the nonreduced components of B. In particular, if ω is a $\mathbb{Q}$-divisor, we may assume that B is a $\mathbb{Q}$-divisor.

Proof of (iv). We check the invariance of the structure under permissible blow-ups (for embedded log transformations, this is easier). The blow-ups do not introduce new (Y, B)-strata; therefore, we only need to check the invariance of the ideal sheaf in (2) of Definition 4.1. By cohomological descent, we may assume that Y is nonsingular and B_Y is a divisor with normal crossing support. Assume that $\sigma\colon (Y', B_{Y'}) \to (Y, B_Y)$ is a crepant log nonsingular model. Denote $\Delta = B_Y - \lfloor B_Y \rfloor$, let R be the reduced part of B_Y, and define Δ' and R' similarly. Note the identity

$$\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor = \lceil -B_Y \rceil + R.$$

We have $(\lceil -B_{Y'}\rceil + R) - \sigma^*(\lceil -B_{Y'}\rceil + R) = K_{Y'} + \Delta' + R' - f^*(K_Y + \Delta + R)$. It is enough to show that the right-hand side is effective. Assume that it is negative in some divisor E. Its coefficient $\mathrm{mult}_E(\Delta' + R') + a(E; \Delta + R) - 1$ is integral; hence, $\mathrm{mult}_E(\Delta' + R') + a(E; \Delta + R) \le 0$ (here $a(E; \Delta + R)$ is the log discrepancy of E with respect to $(Y, \Delta + R)$). Therefore, $\mathrm{mult}_E(\Delta' + R') = 0$ and $a(E; \Delta + R) = 0$. The latter implies that $c_Y(E)$ is a stratum of R; hence, we also have $a(E; B_Y) = 0$ by the normal crossing assumption. Equivalently, $\mathrm{mult}_E(R') = 1$. Contradiction. $\square$

Examples 4.3. 1. Any generalized log variety (X, B) is a quasi-log variety: let ω be any $\mathbb{R}$-Cartier divisor such that $\omega \sim_{\mathbb{R}} K + B$, and let $X_{-\infty}$ be the locus where (X, B) does not have log canonical singularities (with the induced closed subscheme structure). A quasi-log resolution is a log resolution. The qlc centers are exactly the subvarieties C of X such that (X, B) has zero log discrepancy in the generic point of C. With the exception of X (which is a qlc center), the qlc centers of (X, ω) are exactly the *lc centers* of Y. Kawamata [8] that are not included in $(X, B)_{-\infty}$. This is natural since we do not expect any adjunction on lc centers along which (X, B) does not have log canonical singularities.

Conversely, if Y is nonsingular, f is birational, and X is normal, then X is associated (equivalent) to a generalized log variety as above. Indeed, the corresponding generalized log variety is (X, f_*B_Y).

2. Let (Y, B_Y) be a proper log variety such that $K_Y + B_Y$ is nef. The Abundance Conjecture predicts the existence of a proper morphism $f\colon Y \to X$ to a projective variety X such that $K_Y + B_Y \sim_{\mathbb{R}} f^*H$ for some ample divisor $H \in \mathrm{Div}(X)_{\mathbb{R}}$. Then X is a quasi-log variety with qlog canonical singularities, with $\omega \sim_{\mathbb{R}} H$ and quasi-log resolution f.

3. Let $(\overline{X}, \overline{B})$ be a generalized log variety, and assume that $X = \mathrm{LCS}(\overline{X}, \overline{B})$ intersects the open subset on which $(\overline{X}, \overline{B})$ has log canonical singularities. Then X is a quasi-log variety, where $\omega \sim_{\mathbb{R}} (K_{\overline{X}} + \overline{B})|_X$ and $X_{-\infty} = (\overline{X}, \overline{B})_{-\infty}$. A quasi-log resolution of X is induced by restriction to the reduced part of the boundary on a log resolution of $(\overline{X}, \overline{B})$:

$$\begin{array}{ccc} (Y, B_Y) & \longrightarrow & (\overline{Y}, \overline{B}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mu} \\ X & \longrightarrow & (\overline{X}, \overline{B}) \end{array}$$

Here $K_{\overline{Y}} + \overline{B} = \mu^*(K_{\overline{X}} + \overline{B})$, Y is the reduced part of $\overline{B}$, and $B_Y = (\overline{B} - Y)|_Y$.

4. Let X be a divisor with normal crossing support in a nonsingular variety $\overline{X}$, and assume that Y, the reduced part of X, is nonempty. Then X is a quasi-log variety, where $\omega \sim_{\mathbb{R}} (K_{\overline{X}} + X)|_X$, and $X_{-\infty}$ is the union of nonreduced components of X. A quasi-log resolution is $f\colon (Y, B_Y) \to X$, where B_Y is defined by the adjunction formula $K_Y + B_Y = (K_{\overline{X}} + X)|_Y$.

Theorem 4.4 (adjunction & vanishing)**.** *Let X be a quasi-log variety, and let X' be the union of $X_{-\infty}$ with a (possibly empty) union of some qlc centers of X.*

(i) *Assume that $X' \neq X_{-\infty}$. Then X' is a quasi-log variety, with $\omega' = \omega|_{X'}$ and $X'_{-\infty} = X_{-\infty}$. Moreover, the qlc centers of X' are exactly the qlc centers of X that are included in X'.*

(ii) *Assume that X/S is projective, and let $L \in \mathrm{Div}(X)$ such that $L - \omega$ is π-ample. Then $\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)$ is π_*-acyclic.*

Proof. (i) After embedded log transformations, we may assume that the union of all strata of (Y, B_Y) mapped into X', which we denote by Y', is a union of irreducible components of Y. Define $B_{Y'}$ by $(K_Y + B_Y)|_{Y'} = K_{Y'} + B_{Y'}$. We claim that $f\colon (Y', B_{Y'}) \to X'$ is a quasi-log resolution. The adjunction formula is clear; therefore, we just check the second property. Denote $A = \lceil -(B_Y^{<1})\rceil$ and $N = \lfloor B_Y^{>1}\rfloor$. Let Y'' be the subscheme of Y whose ideal sheaf $\mathcal{I}$ is defined by the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_Y(-N) \to \mathcal{O}_{Y'}(-N) \to 0.$$

The ideal of the subscheme X' is the unique ideal sheaf $\mathcal{I}_{X'} \subset \mathcal{I}_{X_{-\infty}}$ for which the induced map $\mathcal{I}_{X'} \to f_*\mathcal{I}(A)$ is an isomorphism. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & f_*\mathcal{I}(A) & \longrightarrow & f_*\mathcal{O}_Y(A-N) & \longrightarrow & f_*\mathcal{O}_{Y'}(A-N) & \longrightarrow & 0 \\
 & & \big\downarrow{=} & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & f_*\mathcal{I}(A) & \longrightarrow & f_*\mathcal{O}_Y(A) & \longrightarrow & f_*\mathcal{O}_{Y''}(A) & & \\
 & & \big\uparrow{\simeq} & & \big\uparrow & & \big\uparrow & & \\
0 & \longrightarrow & \mathcal{I}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & 0
\end{array}$$

The map $f_*\mathcal{O}_{Y'}(A-N) \to f_*\mathcal{O}_{Y''}(A)$ is injective by the definition of $\mathcal{I}$. Moreover, $\mathcal{I}(A) \simeq \mathcal{I}_{Y'} \otimes \mathcal{O}_Y(A-N)$ and $K_Y + B_Y \sim_{\mathbb{R}} 0/X$. By the choice of Y', we deduce from Theorem 3.2(i) that any local section of $R^1 f_*\mathcal{I}(A)$ that is supported by $f(Y')$ is zero. Therefore, the top row is exact. It is easy to see that $\mathcal{I}_{X'_{-\infty}} := \mathcal{I}_{X_{-\infty}}/\mathcal{I} \to f_*\mathcal{O}_{Y'}(A-N)$ is an isomorphism. Finally, the characterization of the qlc centers of X' follows from the choice of Y' and the corresponding statement for $(Y', B_{Y'})$ and (Y, B_Y).

(ii) As in the proof of Theorem 3.2(ii), b), it follows from Theorem 3.2(ii) that $f_*\mathcal{I}(A) \otimes \mathcal{O}_X(L)$ is π_*-acyclic. $\square$

Remark 4.5. The above proof gives a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}_{X'} & \longrightarrow & \mathcal{I}_{X_{-\infty}} & \longrightarrow & \mathcal{I}_{X'_{-\infty}} & \longrightarrow & 0 \\
 & & {=}\big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & 0
\end{array}$$

Therefore, we can lift the global sections of $\mathcal{O}_X(L)$ or $\mathcal{I}_{X_{-\infty}} \otimes \mathcal{O}_X(L)$ from X'/S to X/S.

Definition 4.6. The *LCS locus* of a quasi-log variety X, denoted by $\mathrm{LCS}(X)$, is the union of $X_{-\infty}$ with all qlc centers of X that are not maximal with respect to the inclusion. The subscheme structure is defined as above, and we have a natural embedding $X_{-\infty} \subseteq \mathrm{LCS}(X)$.

Proposition 4.7. *Let X be a quasi-log variety whose LCS locus is empty. Then X is normal.*

Proof. We may assume that X is connected. Let $f\colon (Y, B_Y) \to X$ be a quasi-log resolution of X. By assumption, B_Y is a subboundary, f is surjective with connected fibers, and each stratum of (Y, B_Y) dominates some irreducible component of X. We first show that X is irreducible. Indeed, let $\{X_i\}$ be the irreducible components of X, and let Y_i be the union of strata of Y that dominate X_i. A nonempty intersection of two strata mapped on different components cannot dominate some component of X; thus, Y is the disjoint union of the closed subsets Y_i. But Y is connected since f has connected fibers; thus, X is irreducible.

Let $f_n\colon Y_n \to X$ be the induced morphisms. Then $Y_n = \bigsqcup_j Y_n^j$ is the disjoint union of its irreducible components, and $f_n = \bigsqcup f_n^j$. Each $f_n^j\colon Y_n^j \to X$ is dominant and, thus, factors through the normalization: $f_n^j = \nu \circ g_n^j$. The maps $\{g_n = \bigsqcup g_n^j\}_n$ glue to a morphism $g\colon Y_\bullet \to X^\nu$ that factors $f\colon Y_\bullet \to X$. This map extends to Y according to Lemma 2.2(ii).

Therefore, f factors through the normalization of X. Since f has connected fibers and X is seminormal, the normalization is an isomorphism. $\square$

The following properties of qlc centers generalize [8, 1.5, 1.6] (in particular, minimal lc centers of log varieties have normal singularities):

Proposition 4.8. *Assume that X is a quasi-log variety with qlog canonical singularities. The following hold*:

(i) *The intersection of two qlc centers is a union of qlc centers.*

(ii) *For any point $P \in X$, the set of all qlc centers passing through P has a unique minimal element W. Moreover, W is normal at P.*

Proof. (i) Let C_1 and C_2 be two qlc centers of X. Fixing $P \in C_1 \cap C_2$, it is enough to find a qlc center C such that $P \in C \subset C_1 \cap C_2$. The union $X' = C_1 \cup C_2$ is a quasi-log variety with two irreducible components; hence, it is not normal at P. By Proposition 4.7, $P \in \mathrm{LCS}(X')$. Therefore, there exists a qlc center $C \subset C_1$ with $\dim C < \dim C_1$ such that $P \in C \cap C_2$. If $C \subset C_2$, we are done. Otherwise, we repeat the argument with $C_1 := C$ and reach the conclusion in a finite number of steps.

(ii) The uniqueness follows from (i), and the normality follows from Proposition 4.7. □

Theorem 4.9 (cf. [9])**.** *Let $(X/S, B)$ be a relative generalized log variety. Let $\nu\colon W \to X$ be the normalization of an irreducible component of* $\mathrm{LCS}(X, B)$, *and assume that $\nu(W)$ is an exceptional lc center. The following hold*:

(i) *There exists a quasi-log structure on W such that $\omega \sim_{\mathbb{R}} \nu^*(K + B)$ and* $\mathrm{LCS}(W, \omega) \subseteq \nu^{-1}((X, B)_{-\infty} \cup \bigcup\{C \text{ lc center} \neq \nu(W)\})$.

(ii) *Assume that H is a nef and big $\mathbb{R}$-divisor on W/S. Then there exists a generalized log variety structure (W, B_W) on W such that $\omega + H \sim_{\mathbb{R}} K_W + B_W$ and* $\mathrm{LCS}(W, B_W) \subseteq \mathrm{LCS}(W, \omega)$.

Remarks 4.10. 1. This is a weak form of adjunction. We expect that the inclusion in (i) is an equality (we prove this on a big open subset of W). Furthermore, (ii) should hold in a stronger form: the quasi-log structure of (W, ω) is equivalent to the log structure of (W, B_W).

2. (X, B) induces a natural $\mathbb{R}$-b-divisor $\mathcal{B}_{\mathrm{div}}$ of W, called the *divisorial part of adjunction* (cf. [1, §3]), and the following inequality is expected to hold:

$$\mathcal{A}(W, B_W) \leq -\mathcal{B}_{\mathrm{div}}.$$

If $\dim X \leq 4$, this follows from [14]: there exists a birational model W'/W such that $-\mathcal{B}_{\mathrm{div}} = \mathcal{A}(W', (\mathcal{B}_{\mathrm{div}})_{W'})$. This implies the desired inequality if we choose a high enough model W'/W in step (ii) of the proof.

Proof of Theorem 4.9. (i) The lc center being exceptional means that, among the valuations centered at $\nu(W)$ on X, there exists a unique valuation E with zero log discrepancy with respect to (X, B). Let $\mu\colon (Y, B_Y) \to (X, B)$ be a crepant log resolution such that E is a divisor on Y. We can write $B_Y = E + B'$ and set $B_E = B'|_E$ and $\omega = \nu^*(K + B)$. Since $f\colon E \to \nu(W)$ has connected general fiber, its Stein factorization is $g\colon E \to W$:

$$\begin{tikzcd} (Y, B_Y) \arrow[d, "\mu"'] & (E, B_E) \arrow[l] \arrow[d, "g"] \\ (X, B) & (W, \omega) \arrow[l, "\nu"] \end{tikzcd}$$

We claim that g defines a quasi-log structure on W. Indeed, the crepant hypothesis is satisfied since $g^*\omega \sim_{\mathbb{R}} K_E + B_E$. For the second hypothesis, it suffices to show the following equality:

$$\mathcal{O}_W = g_*\mathcal{O}_E(\lceil -(B_E^{<1}) \rceil).$$

We have a natural inclusion $j\colon \mathcal{O}_W \to g_*\mathcal{O}_E(\lceil -(B_E^{<1}) \rceil)$ which is an isomorphism in the generic point of W. Since $\mathcal{O}_W$ is reflexive and $g_*\mathcal{O}_E(\lceil -(B_E^{<1}) \rceil)$ is torsion free, it is enough to check surjectivity in codimension-one points of W (cf. [15, 2.iv]). For this, we may assume that W is a curve and X is a germ at a closed point $P \in \nu(W)$. If $\lceil -B' \rceil$ is effective, then $\nu(W)$ is normal at P and the desired equality holds. If $\lceil -B' \rceil$ is not effective, then $f_*\mathcal{O}_Y(\lceil -B' \rceil) \subseteq m_{P,X}$. On the other hand, $R^1\mu_*\mathcal{O}_Y(\lceil -B_Y \rceil)$ is torsion free by Theorem 3.2(i). Therefore, we have a surjection

$$\mu_*\mathcal{O}_Y(\lceil -B' \rceil) \to g_*\mathcal{O}_E(\lceil -B_E \rceil) \to 0.$$

In particular, $g_*\mathcal{O}_E(\lceil -B_E \rceil) \subset m_{Q,W}$ for every point $Q \in \nu^{-1}(P)$. This implies that $\lceil -(B_E^{<1}) \rceil$ contains none of the fibers $g^{-1}(Q)$ in its support. Consequently, $\mathcal{O}_W = g_*\mathcal{O}_E(\lceil -(B_E^{<1}) \rceil)$ at P.

By construction, $\nu(\mathrm{LCS}(W,\omega))$ is contained in the union of $(X,B)_{-\infty}$ and all lc centers of (X,B) different from $\nu(W)$ (this is the subscheme of X with ideal sheaf $\mu_*\mathcal{O}_Y(\lceil -B' \rceil)$).

(ii) We may assume that g factors as $g = \sigma \circ h$, where $\sigma\colon W' \to W$ is a resolution such that $(E,P) \xrightarrow{h} (W',Q) \to S$ satisfies the assumptions of Theorem 4.11, B_E is supported by P, $\mathrm{Supp}(B_E^h)$ has relative normal crossings over $W' \setminus Q$, and $h(\mathrm{Supp}(B_E^v)) \subseteq Q$.

Define $B_{W'} = \sum b_i Q_i$ by the formulas $1 - b_i = \min_{P_j/Q_i} \frac{1-b_j}{m_{P_j/Q_i}}$, and let M be an $\mathbb{R}$-divisor on W' such that

$$K_E + B_E \sim_{\mathbb{R}} h^*(K_{W'} + B_{W'} + M).$$

Since $g_*\mathcal{O}_E(\lceil -B_E \rceil) \subset \mathcal{O}_W$, the negative part of $B_{W'}$ is exceptional over W. Also, $\mathrm{LCS}(W', B_{W'}) \subset \sigma^{-1}(\mathrm{LCS}(W,\omega))$: if $b_i \ge 1$, there exists P_j/Q_i such that $b_j \ge 1$; hence, $\sigma(Q_i) = g(P_j) \subset \mathrm{LCS}(W,\omega)$. Note that $B_{\mathrm{div}} = \sigma_* B_{W'}$ is the divisorial part of adjunction induced by (X,B) on W (cf. [1, §3]).

Since $D = B_E - h^*B_{W'}$ satisfies the hypothesis of Theorem 4.11, M is nef$/S$. In particular, $M + \sigma^*H$ is nef and big$/S$; thus, there exists an effective $\mathbb{R}$-divisor Δ with arbitrary small coefficients such that $M + \sigma^*H \sim_{\mathbb{R}} \Delta$. We set $B_W = \sigma_*(B_{W'} + \Delta) = B_{\mathrm{div}} + \sigma_*\Delta$. Then $\sigma\colon (W', B_{W'} + \Delta) \to (W, B_W)$ is a crepant birational contraction; hence, the claim follows. □

Theorem 4.11 [9, Theorem 1]. *Let $h\colon (Y,P) \to (X,Q)$ be a projective contraction of nonsingular quasiprojective varieties endowed with simple normal crossing boundaries $Q = \sum Q_i$ and $P = \sum P_j$, such that h is smooth over $X \setminus Q$, $h^{-1}(Q) \subset P$, and if we decompose $P = P^h + P^v$, then $h(P^v) \subset Q$ and P^h/X has relative simple normal crossings over $X \setminus Q$.*

Assume that X/S is a proper morphism, and let D be an $\mathbb{R}$-divisor on Y with the following properties:

(1) *$K_Y + D \sim_{\mathbb{R}} h^*(K_X + M)$ for some $\mathbb{R}$-divisor M on X.*

(2) *$\lceil -D^h \rceil$ is effective, and $\mathrm{rank}\, h_*\mathcal{O}_Y(\lceil -D \rceil) = 1$.*

(3) *$D = \sum d_j P_j$ is supported by P.*

(4) *For each i, $d_j \le 1 - \mathrm{mult}_{Q_i} h^*P_j$ if $h(P_j) = Q_i$, and equality holds for some j.*

Then M is nef$/S$.

Proof. Assume first that D and M are $\mathbb{Q}$-divisors and $K_Y + D \sim_{\mathbb{Q}} h^*(K_X + M)$. The claim is just a noncompact version of [9, Theorem 1]. The same argument works since the semipositivity follows from local analytic computations. Note that Theorem 1 in [9] is stated under the extra assumption $\lceil -D \rceil \ge 0$, which is however not used during the proof (cf. [1, 3.5]).

Consider the general case. Assumption (1) reads as

$$K_Y + D + \sum_k r_k(\varphi_k) = h^*(K_X + M),$$

where φ_k are rational functions on Y and r_k are real numbers. If we consider D as a function of the r_k's and the coefficients of M, properties (2)–(4) impose rational constraints on them. Therefore, there exist rational approximations $\lim_{l\to\infty} r_k^{(l)} = r_k$, $\lim_{l\to\infty} D^{(l)} = D$, and $\lim_{l\to\infty} M^{(l)} = M$ such that

$$K_Y + D^{(l)} + \sum_k r_k^{(l)}(\varphi_k) = h^*(K_X + M^{(l)}),$$

$D^{(l)}$ satisfies (2)–(4), and $D^{(l)} - D$ and $M^{(l)} - M$ are supported by the irrational components of D and M, respectively. The rational case implies that $M^{(l)}$ is nef$/S$ for every l. But $M = \lim_{l\to\infty} M^{(l)}$; hence, M is nef$/S$ as well. $\square$

5. THE CONE THEOREM

We follow the arguments of [10, 2-4] and [13], which we also refer to for references.

Theorem 5.1 (Base Point Free Theorem)**.** *Assume that X/S is a projective quasi-log variety. Let L be a π-nef Cartier divisor on X such that*

(i) *$qL - \omega$ is a π-ample for some $q \in \mathbb{R}$;*

(ii) *$\mathcal{O}_{X_{-\infty}}(mL)$ is $\pi|_{X_{-\infty}}$-generated for $m \gg 0$.*

Then $\mathcal{O}_X(mL)$ is π-generated for $m \gg 0$.

Proof. We may shrink S to an affine open subset without further notice.

1. $\mathcal{O}_X(mL)$ is π-generated on $\mathrm{LCS}(X)$ for $m \gg 0$. Set $X' = \mathrm{LCS}(X)$. The vanishing $R^1\pi_*\mathcal{I}_{X'} \otimes \mathcal{O}_X(mL) = 0$ $(m \geq q)$ implies the surjectivity of the top horizontal map in the diagram below:

$$\begin{array}{ccc} \pi^*\pi_*\mathcal{O}_X(mL) & \longrightarrow & \pi^*\pi_*\mathcal{O}_{X'}(mL) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha'} \\ \mathcal{O}_X(mL) & \longrightarrow & \mathcal{O}_{X'}(mL) \end{array}$$

If $X' = X_{-\infty}$, α' is surjective for $m \gg 0$ by assumption. If $X' \neq X_{-\infty}$, then X' is a quasi-log variety; hence, α' is surjective for $m \gg 0$ by induction. Therefore, α is surjective on X' for $m \gg 0$.

2. $\mathcal{O}_X(mL)$ is π-generated on a nonempty set for $m \gg 0$. According to step 1, we may assume that $\mathrm{LCS}(X) = \varnothing$. In particular, X is normal.

(a) Assume that L is π-numerically trivial. Vanishing implies that $\pi_*\mathcal{O}_X(L)$ and $\pi_*\mathcal{O}_X(-L)$ are nonzero [17]. Therefore, L is trivial and, hence, π-generated.

(b) Assume that L is not π-numerically trivial. Denote $H = qL - \omega$. Using a quasi-log resolution of X, we can find an $\mathbb{R}$-divisor D on X such that $D \sim_{\mathbb{Q}} c(H + mL)$, $0 < c < 1$, and $(X, \omega + D)$ has qlog canonical singularities, with nonempty LCS locus [17]. Setting $q' = q + cm$, we reduce the case to step 1.

3. Assume that $\mathcal{O}_X(mL)$ is π-generated on a nonempty subset containing $\mathrm{LCS}(X)$ and denote by $\mathrm{Bsl}_\pi |mL|$ the locus X where $\mathcal{O}_X(mL)$ is not π-generated. Then $\mathrm{Bsl}_\pi |mL|$ is not contained in $\mathrm{Bsl}_\pi |m'L|$ for $m' \gg 0$.

Let $f\colon (Y, B) \to X$ be a quasi-log resolution. For $D \in |mL|$ general, we may assume that $f^*D = F + M$ has multicrossing support with respect to (Y, B_Y), where F is the π-fixed part and M is reduced. Let c be maximal such that $B'_Y = B_Y + cf^*D$ is a subboundary above $X \setminus X_{-\infty}$. Then $f\colon (Y, B'_Y) \to (X, \omega')$ is a quasi-log resolution of a quasi-log variety, with $\omega' = \omega + cD$ and $X'_{-\infty} = X_{-\infty}$. Moreover, (X, ω') has a qlc center C included in $\mathrm{Bsl}_\pi |mL|$. Applying step 1 with $q' = q + cm$, we infer that $\mathcal{O}_X(m'L)$ is π-generated on C for $m' \gg 0$.

4. The above steps imply that $\mathcal{O}_X(aL)$ and $\mathcal{O}_X(bL)$ are π-generated if a and b are very high powers of two prime numbers. Since a and b are relatively prime, they generate the semigroup $\mathbb{Z}_{\geq N}$ for some N. Therefore, $\mathcal{O}_X(mL)$ is π-generated for $m \geq N$. $\square$

Definition 5.2. Let $(X/S, \omega)$ be a quasi-log variety, with non-qlog canonical locus $X_{-\infty}$. Set

$$\overline{\mathrm{NE}}(X/S)_{-\infty} := \mathrm{Im}(\overline{\mathrm{NE}}(X_{-\infty}/S) \to \overline{\mathrm{NE}}(X/S)).$$

For $D \in \mathrm{Div}(X)_{\mathbb{R}}$, set $D_{\geq 0} := \{z \in N_1(X/S);\ D \cdot z \geq 0\}$ (similarly for > 0, ≤ 0, and < 0) and $D^{\perp} := \{z \in N_1(X/S);\ D \cdot z = 0\}$. We also use the notation

$$\overline{\mathrm{NE}}(X/S)_{D \geq 0} := \overline{\mathrm{NE}}(X/S) \cap D_{\geq 0},$$

and similarly for > 0, ≤ 0, and < 0.

Definition 5.3. An *extremal face* of $\overline{\mathrm{NE}}(X/S)$ is a nonzero subcone $F \subseteq \overline{\mathrm{NE}}(X/S)$ such that $z, z' \in \overline{\mathrm{NE}}(X/S)$ and $z + z' \in \overline{\mathrm{NE}}(X/S)$ imply that $z, z' \in F$. Equivalently, $F = \overline{\mathrm{NE}}(X/S) \cap H^{\perp}$ for some π-nef $\mathbb{R}$-divisor $H \in \mathrm{Div}(X)_{\mathbb{R}}$ (called a *supporting function of* F). An *extremal ray* is a 1-dimensional extremal face.

(i) An extremal face F is called *ω-negative* if $F \cap \overline{\mathrm{NE}}(X/S)_{\omega \geq 0} = \{0\}$.

(ii) An extremal face F is called *relatively ample at infinity* if $F \cap \overline{\mathrm{NE}}(X/S)_{-\infty} = \{0\}$. Equivalently, $H|_{X_{-\infty}}$ is $\pi|_{X_{-\infty}}$-ample for any supporting function $H \in \mathrm{Div}(X)_{\mathbb{R}}$ of F.

(iii) An extremal face F is called *contractible at infinity* if it has a rational supporting function $H \in \mathrm{Div}(X)_{\mathbb{Q}}$ such that $H|_{X_{-\infty}}$ is $\pi|_{X_{-\infty}}$-semiample.

Remarks 5.4. 1. Let F be an extremal face that is ample at infinity. Then F is contractible at infinity if and only if F is rational, i.e., if it has a supporting function given by a rational divisor. We will show in the Cone Theorem that if an ω-negative extremal face is ample at infinity, then it is contractible at infinity.

2. Any ω-negative extremal face is relatively ample at infinity if ω is relatively nef on $X_{-\infty}$ (in particular, if $X_{-\infty}$ is empty).

Definition 5.5. Let F be an extremal face of $\overline{\mathrm{NE}}(X/S)$. The *contraction of* F is a projective morphism onto a projective variety Y/S

$$\begin{tikzcd} X \arrow[rr, "\varphi_F"] \arrow[dr, "\pi"'] & & Y \arrow[dl, "\sigma"] \\ & S & \end{tikzcd}$$

satisfying the following properties:

(1) Let C be an irreducible curve of X such that $\pi(C)$ is a point. Then $\varphi_F(C)$ is a point if and only if $[C] \in F$.

(2) $\mathcal{O}_Y = (\varphi_F)_* \mathcal{O}_X$.

By Zariski's Main Theorem, such a morphism is unique if it exists.

Theorem 5.6 (Contraction Theorem)**.** *Let X/S be a projective quasi-log variety. Let F be an ω-negative extremal face of $\overline{\mathrm{NE}}(X/S)$ that is contractible at infinity. Then the contraction of the face F exists.*

Proof. Let $H \in \mathrm{Div}(X)$ be a π-nef divisor such that $H|_{X_{-\infty}}$ is relatively semiample and $F = \overline{\mathrm{NE}}(X/S) \cap H^{\perp}$. By Kleiman's ampleness criterion, $aH - \omega$ is π-ample for some positive integer a. Scaling H, we may assume that its restriction at infinity is relatively free. According to the Base Point Free Theorem, some multiple of H is relatively free. The Stein factorization $\varphi\colon X/S \to Y/S$ of the associated morphism satisfies the following properties:

(1) $H \sim_{\mathbb{Q}} \varphi^*(A)$ for some relatively ample $A \in \mathrm{Div}(Y)_{\mathbb{Q}}$.

(2) $\mathcal{O}_Y = \varphi_* \mathcal{O}_X$.

Since A is relatively ample, it is clear that φ is the contraction of the face F. $\square$

Remark 5.7. Let F be an ω-negative extremal face that is contractible at infinity. Then F is relatively ample at infinity if and only if the associated contraction $\varphi_F\colon X \to Y$ embeds $X_{-\infty}$ into Y.

Lemma 5.8. *Let $P(x,y)$ be a nontrivial polynomial of degree at most d, let a be a positive integer, and let r be either an irrational number or a rational number such that, in reduced form, ra has numerator bigger than $(d+1)a$. Then $P(x,y) \neq 0$ for all sufficiently large integral points in the strip $\{rax - r < y < rax\}$.*

Proof. If r is not rational, there are integral points of the strip that are infinitely close to the line $\{y = rax\}$. If r is rational, let $ra = \frac{u}{v}$ be the reduced form decomposition. The line $\{y = rax - \frac{1}{v}\}$ has infinitely many integral points, and it is included in the strip $\{rax - \frac{r}{d+1} < y < rax\}$ if $u > a(d+1)$.

In both cases, there are infinitely many rays passing through the origin that have at least $d+1$ integral points common with the strip $\{rax - r < y < rax\}$. Since P is nontrivial, it cannot vanish on more than a finite number of them. $\square$

Theorem 5.9 (Rationality Theorem)**.** *Assume that X/S is a projective quasi-log variety such that $\omega \in \mathrm{Div}(X)_{\mathbb{Q}}$. Let H be a π-ample Cartier divisor on X, and let r be a positive number such that*

(i) $\omega + rH$ *is π-nef but not π-ample*;

(ii) $(\omega + rH)|_{X_{-\infty}}$ *is $\pi|_{X_{-\infty}}$-ample.*

Then r is a rational number, and, in reduced form, ra has numerator at most $a(\dim X/S + 1)$, where a is the index of ω.

Proof. Assume, by contradiction, that r does not satisfy the required properties. In particular, the strip

$$\mathcal{S} = \{(x,y) \in \mathbb{N}^2;\ rax - r < y < rax,\ (x,y) \text{ large}\}$$

has infinitely many points. Set $L(x,y) = xa\omega + yH$. The family of Cartier divisors $\{L(x,y)\}_{(x,y)\in\mathcal{S}}$ has the following properties with respect to (X,ω):

(1) The locus $\mathrm{Bsl}_\pi |L(x,y)|$, where $\mathcal{O}_X(L(x,y))$ is not π-generated, is independent of $(x,y) \in \mathcal{S}$. We denote this base locus by Λ.

Proof. Note first that if (x,y) is a given point of $\mathcal{S}$ and (kx,ky) is a large multiple that does not lie in $\mathcal{S}$, then $L(x',y') - L(kx,ky)$ is π-ample and π-generated for large $(x',y') \in \mathcal{S}$. In particular, for a given (x,y), $\mathrm{Bsl}_\pi |L(x,y)|$ contains $\mathrm{Bsl}_\pi |L(x',y')|$ for large $(x',y') \in \mathcal{S}$. The claim follows by Noetherian induction.

(2) $L(x,y)$ is an adjoint divisor with respect to ω for all (x,y).

Proof. $L(x,y) - \omega = (xa-1)(\omega + rH) + (y - rax + r)H$ is π-ample for $y > rax - r$. Note that $L(x,y)$ is π-ample for $y > rax$.

(3) $\Lambda \cap (X,\omega)_{-\infty} = \varnothing$, and, for each qlc center C of (X,ω), there exists (x,y) such that $\mathcal{O}_C(L(x,y))$ is $\pi|_C$-generated on some nonempty subset.

Proof. Since $L(x,y)$ are adjoint with respect to ω, we can lift the global sections of $\mathcal{O}_X(L(x,y))$ from $X_{-\infty}$. Therefore, Λ does not intersect the non-qlog canonical locus if $\mathcal{O}_{X_{-\infty}}(L(x,y))$ is relatively generated for infinitely many values in $\mathcal{S}$. The line $y = rax$ is relatively ample on $X_{-\infty}$; hence, Lemma 5.8 implies the existence of infinitely many points (x,y) of $\mathcal{S}$ for which $L(x,y)|_{X_{-\infty}}$ is relatively ample. The same argument as in (1) shows that $\mathcal{O}_{X_{-\infty}}(L(x,y))$ are relatively generated for large values.

For the latter part, let C be a qlc center of X. We may assume that C does not intersect $X_{-\infty}$ and S is a point. By adjunction, $L(x,y)|_C$ are adjoint; hence,

$$P(x,y)=\dim H^0(C,\mathcal{O}_C(L(x,y)))=\chi(C,\mathcal{O}_C(L(x,y)))$$

is a polynomial of degree at most $\dim C\le\dim X/S$. It is a nontrivial polynomial; hence, $P(x,y)\neq 0$ for $(x,y)\in\mathcal{S}$ by Lemma 5.8 again.

By adjunction, for any family $L(x,y)$ satisfying (1)–(3) above, the common base locus Λ does not intersect $X_{-\infty}$ and does not contain any qlc center of X.

If $\Lambda=\varnothing$, then $\mathcal{O}_X(L(x,y))$ is π-generated, in particular, π-nef. This is a contradiction. Therefore, Λ is nonempty. Let D be a general member of $|L(x,y)|$, and choose $0<c\le 1$ maximal such that $\omega':=\omega+cD$ has qlog canonical singularities outside $X_{-\infty}$. Note that (X,ω') and (X,ω) have the same non-qlog canonical locus and (X,ω') has a qlc center contained in Λ. But $\{L(x,y)\}_{(x,y)\in\mathcal{S}}$ has the same properties (1)–(3) with respect to (X,ω'); hence, Λ cannot contain any qlc center of (X,ω'). Contradiction. □

Theorem 5.10 (Cone Theorem)**.** *Let $(X/S,\omega)$ be a projective quasi-log variety. Let $\{R_j\}$ be the ω-negative extremal rays of $\overline{\mathrm{NE}}(X/S)$ that are relatively ample at infinity. Then the following hold*:

(i) $\overline{\mathrm{NE}}(X/S)=\overline{\mathrm{NE}}(X/S)_{\omega\ge 0}+\overline{\mathrm{NE}}(X/S)_{-\infty}+\sum R_j$.

(ii) *There are only finitely many R_j's included in $(\omega+H)_{<0}$, for any relatively ample $H\in\mathrm{Div}(X)_{\mathbb{R}}$. In particular, the R_j's are discrete in the half-space $\omega_{<0}$.*

(iii) *Let F be an ω-negative extremal face of $\overline{\mathrm{NE}}(X/S)$ that is relatively ample at infinity. Then F is a rational face* (*in particular, contractible at infinity*).

Proof. Assume first that $\omega\in\mathrm{Div}(X)_{\mathbb{Q}}$.

(1) If $\dim_{\mathbb{R}}N_1(X/S)\ge 2$, then

$$\overline{\mathrm{NE}}(X/S)=\overline{\mathrm{NE}}(X/S)_{\omega\ge 0}+\overline{\mathrm{NE}}(X/S)_{-\infty}+\overline{\sum_F F},$$

where the F's vary among all rational proper ω-negative extremal faces that are relatively ample at infinity, and the overline denotes the closure with respect to the real topology.

Proof. Denote the right-hand side by B. If equality does not hold, there exists a separating function $M\in\mathrm{Div}(X)\setminus\{0\}$, which is not a multiple of ω in $N^1(X/S)$, such that M is positive on $B\setminus\{0\}$ but is not relatively nef. Since M belongs to the interior of the dual cone of $\overline{\mathrm{NE}}(X/S)_{\omega\ge 0}$, we can scale it so that $M=\omega+H$ for a relatively ample $\mathbb{Q}$-Cartier divisor H.

Let $r>1$ be the largest real number such that $\omega+rH$ is relatively nef but not ample. In particular, $\omega+rH$ is relatively ample on $X_{-\infty}$. By the Rationality and Contraction Theorems, r is a rational number and the extremal face $F\neq\{0\}$ with the supporting function $\omega+rH$ can be contracted. If F is proper, it is contained in B; hence, M is relatively ample on F. This contradicts $r>1$. Otherwise, $\omega+rH$ is trivial and $M=\frac{r-1}{r}\omega$ in $N^1(X/S)$, which contradicts the choice of M.

(2) We may take only proper rays in (1).

Proof. Let F be a rational proper ω-negative extremal face that is relatively ample at infinity, and assume that $\dim F\ge 2$. Let $\varphi_F\colon X\to W$ be the associated contraction, so that $-\omega$ is φ_F-ample. Applying (1) to X/W, we obtain

$$F=\overline{\mathrm{NE}}(X/W)\setminus\{0\}=\left(\overline{\mathrm{NE}}(X/W)_{-\infty}+\overline{\sum_G G}\right)\setminus\{0\},$$

where the G's are the rational proper ω-negative extremal faces of $\overline{\mathrm{NE}}(X/W)$ that are relatively ample at infinity. Since φ_F embeds $X_{-\infty}$ into W, $\overline{\mathrm{NE}}(X/W)_{-\infty}=0$. The G's are also ω-negative extremal faces of $\overline{\mathrm{NE}}(X/S)$ that are contractible at infinity, and $\dim G<\dim F$. By induction, we obtain

$$\overline{\mathrm{NE}}(X/S)=\overline{\mathrm{NE}}(X/S)_{\omega\geq 0}+\overline{\mathrm{NE}}(X/S)_{-\infty}+\overline{\sum R_j}.$$

Note that each R_j does not intersect $\overline{\mathrm{NE}}(X/S)_{-\infty}$.

(3) Let A be a relatively ample Cartier divisor on X. Then each R_j is generated by an irreducible reduced curve C_j, $r_j=\frac{A\cdot C_j}{-\omega\cdot C_j}$ is a rational number, and the denominator of $\frac{r_j}{a}$, written in reduced form, is at most $a(d+1)$. Indeed, each R_j is contractible, and the statement follows from the Rationality Theorem applied to the contraction φ_{R_j}.

(4) Let $\{H_i\}_{i=1}^{\varrho-1}$ (where ϱ is the Picard number) be relatively ample Cartier divisors on X that, together with ω, form a basis over $\mathbb{R}$ of $N^1(X/S)$. By (3), $R_j\cap\{z;\ -a\omega\cdot z=1\}$ is included in the lattice

$$\{z;\ -a\omega\cdot z=1,\ H_i\cdot z\in(a(d+1)!)^{-1}\mathbb{Z}\}.$$

Therefore, the extremal rays are discrete in the half-space $\omega_{<0}$, and the real closure can be omitted. We have obtained (i).

(5) We show (ii). Let $H\in\mathrm{Div}(X)_{\mathbb{R}}$ be relatively ample. Since $H-\sum_{i=1}^{\varrho-1}\epsilon_i H_i$ is ample for $0<\epsilon_i\ll 1$, the R_j's included in $(\omega+H)_{<0}$ correspond to some elements of the above lattice for which $\sum_{i=1}^{\varrho-1}\epsilon_i H_i\cdot z<\frac{1}{a}$. They are finite.

(6) We show (iii). The vector space $V=F^{\perp}\subset N^1(X)$ is defined over $\mathbb{Q}$ since F is generated by some of the R_j's. There exists a relatively ample divisor $H\in\mathrm{Div}(X)$ such that $F\subset(\omega+H)_{<0}$. Let $\langle F\rangle$ be the vector space spanned by F, and set

$$W_F=\overline{\mathrm{NE}}(X/S)_{\omega+H\geq 0}+\overline{\mathrm{NE}}(X/S)_{-\infty}+\sum_{R_j\not\subseteq F}R_j.$$

Then W_F is a closed cone, $\overline{\mathrm{NE}}(X/S)=W_F+F$, $W_F\cap\langle F\rangle=\{0\}$, and the supporting functions of F are the elements of V that are positive on $W_F\setminus\{0\}$. This is a nonempty open set and thus contains a rational element that, after scaling, gives a relatively nef Cartier divisor L such that $F=L^{\perp}\cap\overline{\mathrm{NE}}(X/S)$. Therefore, F is rational.

The general case when $\omega\in\mathrm{Div}(X)_{\mathbb{R}}$ can be reduced to the rational case via the following trick: If $H\in\mathrm{Div}(X)_{\mathbb{R}}$ is relatively ample and $\omega+H\in\mathrm{Div}(X)_{\mathbb{Q}}$, we can write $H=E+H'$, where $H'\in\mathrm{Div}(X)_{\mathbb{R}}$ is relatively ample and $(X,\omega':=\omega+E)$ is a quasi-log variety with the same qlc centers and non-qlog canonical locus as (X,ω). Therefore, $\omega+H=\omega'+H'$, $\omega'\in\mathrm{Div}(X)_{\mathbb{Q}}$, and $(X,\omega)_{-\infty}=(X,\omega')_{-\infty}$. In (ii), we may assume that $\omega+H\in\mathrm{Div}(X)_{\mathbb{Q}}$, and in (iii) we can replace ω by $\omega+H\in\mathrm{Div}(X)_{\mathbb{Q}}$. As for (i), we have

$$\overline{\mathrm{NE}}(X/S)=\overline{\mathrm{NE}}(X/S)_{\omega+H\geq 0}+\overline{\mathrm{NE}}(X/S)_{-\infty}+\sum_{(\omega+H)\cdot R_j<0}R_j$$

since the same holds for $\omega'+H'=\omega+H$. Letting H converge to 0, we obtain (i) using (ii). □

Corollary 5.11. *Let X/S be a projective quasi-log variety such that ω is relatively nef on $X_{-\infty}$. If ω is not relatively nef, there exists an ω-negative extremal ray that is relatively ample at infinity.*

6. QUASI-LOG FANO CONTRACTIONS

We specialize the results of the previous section to the equivalent of Fano contractions in our category.

Definition 6.1. A *quasi-log Fano contraction* X/S is a relative projective quasi-log variety X/S such that $-\omega$ is relatively ample and $\mathcal{O}_S = \pi_*\mathcal{O}_X$.

Theorem 6.2. *A projective quasi-log Fano contraction X/S has only finitely many ω-negative extremal rays R_j that are relatively ample at infinity, and $\overline{\mathrm{NE}}(X/S) = \overline{\mathrm{NE}}(X/S)_{-\infty} + \sum R_j$.*

Furthermore, $\mathrm{NE}(X/S)$ is a closed rational polyhedral cone spanned by the R_j's if $X_{-\infty}/S$ has at most finite fibers.

Lemma 6.3. *Assume that $X/T \to S/T$ is a diagram of projective morphisms such that X/S is a quasi-log Fano contraction.*

(i) *There exists an ω-negative extremal face F of $\overline{\mathrm{NE}}(X/T)$ that is contractible at infinity and such that $X/T \to S/T$ is the contraction of the face F.*

(ii) *Let $L \in \mathrm{Div}(X)_K$ such that $L \equiv 0/S$. Then there exists $H \in \mathrm{Div}(S)_K$ such that $L \sim_K \pi^*H$ if one of the following holds*:

$K = \mathbb{Z}$: *$mL|_{X_{-\infty}}$ is relatively base point free for $m \gg 0$.*

$K = \mathbb{Q}$: *$L|_{X_{-\infty}}$ is relatively semiample.*

$K = \mathbb{R}$: *$X_{-\infty}/S$ has at most finite fibers.*

Corollary 6.4. *Let X/S be a quasi-log Fano contraction.*

(i) *Assume that $L \in \mathrm{Div}(X)_{\mathbb{Q}}$ is relatively nef and $L|_{X_{-\infty}}$ is relatively semiample. Then L is relatively semiample.*

(ii) *Assume that $L \in \mathrm{Div}(X)_{\mathbb{R}}$ is relatively nef and $L|_{X_{-\infty}}$ is relatively ample. Then L is relatively semiample.*

Proof. Statement (i) follows from the Base Point Free Theorem. For (ii), assume that $L \in \mathrm{Div}(X)_{\mathbb{R}}$ is relatively nef and $L|_{X_{-\infty}}$ is relatively ample. If $[L] = 0 \in N^1(X/S)$, we just apply Lemma 6.3(ii).

If $[L] \neq 0 \in N^1(X/S)$, $F := L^\perp \cap \overline{\mathrm{NE}}(X/S)$ is a nontrivial face. By assumption, $F \cap (\overline{\mathrm{NE}}(X/S)_{\omega \geq 0} + \overline{\mathrm{NE}}(X/S)_{-\infty}) = \{0\}$. Theorem 5.10(iii) and the Contraction Theorem imply that F is an ω-negative extremal face contractible at infinity and the contraction $\varphi_F \colon X/S \to T/S$ exists. We have $L \equiv 0/T$ and $X_{-\infty}/T$ is an embedding. By Lemma 6.3(ii), $L \sim_{\mathbb{R}} \pi^*H$ for some relatively ample $H \in \mathrm{Div}(T)_{\mathbb{R}}$; i.e., L is relatively semiample. $\square$

Remark 6.5 (cf. Artin's numerical criterion)**.** Let $\pi \colon X \to S$ be a projective birational morphism of normal varieties, and let D be an effective $\mathbb{Q}$-Cartier divisor on X such that the following hold:

- (X, B) is a log variety.
- $-D$ is π-ample.
- For every subscheme $Y \subset X$ supported by $\mathrm{Supp}(D)$, any π-nef Cartier divisor $L \in \mathrm{Div}(Y)$ is π-semiample.

Then any π-nef Cartier divisor L on X is π-semiample. Indeed, $(X/S, B + rD)$ is a quasi-log Fano contraction for $r \gg 0$, with non-log canonical locus supported by $\mathrm{Supp}(D)$. The claim follows from Corollary 6.4(i).

Theorem 6.6. *Let $\pi\colon X \to S$ be a quasi-log Fano contraction, and let $P \in S$ be a closed point.*

(i) *Assume that $X_{-\infty} \cap \pi^{-1}(P) \neq \varnothing$ and C is a qlc center such that $C \cap \pi^{-1}(P) \neq \varnothing$. Then $C \cap X_{-\infty} \cap \pi^{-1}(P) \neq \varnothing$.*

(ii) *Assume that X has qlog canonical singularities. Then the set of all qlc centers intersecting $\pi^{-1}(P)$ has a unique minimal element with respect to inclusion.*

Proof. Let C be a qlc center of X such that $P \in \pi(C) \cap \pi(X_{-\infty})$. By Theorem 4.4 (with $L = 0$), $X' := C \cup X_{-\infty}$ is a quasi-log variety and the restriction map $\pi_*\mathcal{O}_X \to \pi_*\mathcal{O}_{X'}$ is surjective. Since $\mathcal{O}_S = \pi_*\mathcal{O}_X$, $X_{-\infty}$ and C intersect over a neighborhood of P.

Assume now that $X_{-\infty} = \varnothing$, and let C_1, C_2 be two qlc centers of X such that $P \in \pi(C_1) \cap \pi(C_2)$. The union $X' = C_1 \cup C_2$ is a quasi-log variety, and the same argument implies the surjectivity of the restriction map $\pi_*\mathcal{O}_X \to \pi_*\mathcal{O}_{X'}$. Therefore, C_1 and C_2 intersect over P. Furthermore, the intersection $C_1 \cap C_2$ is a union of qlc centers by Proposition 4.8. By induction, there exists a unique qlc center C_P over a neighborhood of P such that $C_P \subseteq C$ for every qlc center C with $P \in \pi(C)$. $\square$

7. THE LOG BIG CASE

For certain applications, we need to weaken the projectivity assumption in the Base Point Free Theorem.

Definition 7.1 (M. Reid)**.** Let X/S be a proper quasi-log variety. A relatively nef $\mathbb{R}$-Cartier divisor H on X is called *log big* if $H|_C$ is relatively big for every qlc center C of X.

Theorem 7.2 (cf. [5, 6])**.** *Let X/S be a proper quasi-log variety, and let L be a relatively nef Cartier divisor on X with the following properties*:

(i) *$qL - \omega$ is relatively nef and log big for some $q \in \mathbb{R}$.*

(ii) *$\mathcal{O}_{X_{-\infty}}(mL)$ is relatively generated for $m \gg 0$.*

Then $\mathcal{O}_X(mL)$ is π-generated for $m \gg 0$.

The proof is parallel to Theorem 5.1. We just need the appropriate equivalent of Theorem 4.4:

Theorem 7.3. *Let X/S be a proper quasi-log variety, and let X' be the union of $X_{-\infty}$ with a union of some qlc centers of X. Let L be a Cartier divisor on X such that $L - \omega$ is relatively nef and log big. Then $\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)$ is π_*-acyclic.*

This is a formal consequence of the log big extension of Theorem 3.2, which we prove below by reduction to the ample case.

Theorem 7.4. *Let $f\colon (Y, B) \to X$ be a proper morphism from an embedded normal crossing pair, such that B is a boundary. Let $L \in \operatorname{Div}(Y)$, let $\pi\colon X \to S$ be a proper morphism, and assume that $L \sim_{\mathbb{R}} K + B + f^*H$ for a nef and log big/S $\mathbb{R}$-Cartier divisor H on X. Then*

(i) *every nonzero local section of $R^q f_*\mathcal{O}_Y(L)$ contains in its support the f-image of some strata of (Y, B);*

(ii) *$R^q f_*\mathcal{O}_Y(L)$ is π_*-acyclic.*

Proof. (1) Assume first that each stratum of (Y, B) dominates some irreducible component of X. Taking the Stein factorization, we may assume that f has connected fibers. Assume then that X is connected, which implies that X is irreducible and each stratum of (Y, B) dominates X. By Chow's lemma, there exists a proper birational morphism $\mu\colon X'/S \to X/S$ such that X'/S is projective. Replacing Y by some blow-up, we may assume that f factors through μ: $f = \mu \circ g$. Set $\mathcal{F} = R^q g_*\mathcal{O}_Y(L)$. Since μ^*H is nef and big over S, and X'/S is projective, we may write

$\mu^*H = E + A$, where E is an effective $\mathbb{R}$-divisor such that $B + g^*E$ has multicrossing support, $\lfloor B \rfloor = \lfloor B + g^*E \rfloor$, and $A \in \mathrm{Div}(X')$ is ample over S. From the ample case, we infer that $\mathcal{F}$ is μ_*- and $(\pi \circ \mu)_*$-acyclic and satisfies (i). Therefore, $R^q f_* \mathcal{O}_Y(L) \simeq \mu_* \mathcal{F}$ satisfies (i) and (ii).

(2) We treat the general case by induction on $\dim X$. We may assume that $Y = Y' \cup Y''$ is a decomposition of Y such that Y' is the union of all strata of (Y, B) that are not mapped to irreducible components of X. Since $f\colon (Y'', B'') \to X$ and L'' satisfy the assumption in (1), the long exact sequence of $0 \to j_* \mathcal{O}_{Y''}(L'') \to \mathcal{O}_Y(L) \to \mathcal{O}_{Y'}(L) \to 0$ with respect to f_* breaks up into short exact sequences

$$0 \to R^q f_* \mathcal{O}_{Y''}(L'') \to R^q f_* \mathcal{O}_Y(L) \to R^q f_* \mathcal{O}_{Y'}(L) \to 0.$$

Since (i) and (ii) hold for the first and third members by case (1) and by induction on dimension, respectively, they also hold for $R^q f_* \mathcal{O}_Y(L)$. $\square$

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