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The set of toric minimal log discrepancies*

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Abstract: We describe the set of minimal log discrepancies of toric log varieties, and study its accumulation points.

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1 Introduction

Minimal log discrepancies are invariants of singularities of log varieties. A log variety (X, B) is a normal variety X endowed with an effective Weil $\mathbb{R}$-divisor B, having at most log canonical singularities. For any Grothendieck point $\eta \in X$, the minimal log discrepancy of (X, B) at η is a non-negative real number denoted $a(\eta; X, B)$. For example, $a(\eta; X, B) = 1 - \operatorname{mult}_\eta(B)$ for every codimension one point $\eta \in X$. For higher codimensional points, minimal log discrepancies can be computed on a suitable resolution of X.

Let $A \subset [0, 1]$ be a set containing 1 and let d be a positive integer. Denote by $\operatorname{Mld}_d(A)$ the set of minimal log discrepancies $a(\eta; X, B)$, where $\eta \in X$ is a Grothendieck point of codimension d, and (X, B) is a log variety whose minimal log discrepancies in codimension one belong to A. For example, $\operatorname{Mld}_1(A) = A$. In connection to the termination of a sequence of log flips (see [8, 10]), Shokurov conjectured that if A satisfies the ascending chain condition, so does $\operatorname{Mld}_d(A)$. Furthermore, under certain assumptions,

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the accumulation points of $\mathrm{Mld}_d(A)$ should correspond to minimal log discrepancies of smaller codimensional points. This is known to hold for $d = 2$ (Shokurov [9], Alexeev [1]) and for any d in the case of toric varieties without boundary (Borisov [3]). The purpose of this note is to extend Borisov's result to the case of toric log varieties. Given the explicit nature of the toric case, we hope this will provide the reader with some interesting examples.

In order to state the main result, define $\mathrm{Mld}_d^{tor}(A) \subset \mathrm{Mld}_d(A)$ as above, except that we further require that X is a toric variety and B is torus invariant. Note that $\mathrm{Mld}_1^{tor}(A) = A$.

Theorem 1.1. *The following properties hold for $d \geq 2$:*

(1) We have

$$\mathrm{Mld}_d^{tor}(A) = \{\sum_{i=1}^s x_i a_i \left| \begin{array}{l} 2 \leq s \leq d \\ (x_1, \dots, x_s) \in \mathbb{Q}^s \cap (0,1]^s, (a_1, \dots, a_s) \in A^s \\ \mathrm{index}(x_i) | \, \mathrm{index}(x_1, \dots, \hat{x}_i, \dots, x_s), \ \forall 1 \leq i \leq s \\ \sum_{i=1}^s (1 + (m-1)x_i - \lceil m x_i \rceil) a_i \geq 0 \ \forall m \in \mathbb{Z} \end{array} \right\},$$

where for a rational point $x \in \mathbb{Q}^n$, we denote by $\mathrm{index}(x)$ *the smallest positive integer q such that $qx \in \mathbb{Z}^n$.*

(2) If A satisfies the ascending chain condition, then so does $\mathrm{Mld}_d^{tor}(A)$.

(3) Assume that A has no nonzero accumulation points. Then the set of accumulation points of $\mathrm{Mld}_d^{tor}(A)$ *is included in*

$$\{0\} \cup \bigcup_{1 \leq d' \leq d-1} \mathrm{Mld}_{d'}^{tor}(\{\frac{1}{n}; n \geq 1\} \cdot A).$$

Equality holds if $d = 2$, or if $\{\frac{1}{n}; n \geq 1\} \cdot A \subseteq A$.

We use the same methods as Borisov [3, 4]. The explicit description in (1) is straightforward, whereas the accumulation behaviour in (2) and (3) relies on a result of Lawrence [6] stating that the set of closed subgroups of a real torus, which do not intersect a given open subset, has finitely many maximal elements with respect to inclusion.

Finally, we should point out that $\mathrm{Mld}_d^{tor}(A)$ is strictly smaller than $\mathrm{Mld}_d(A)$ in general. For example, even the set of accumulation points of $\mathrm{Mld}_2(A)$ (see Shokurov [9] for an explicit description) is larger than $\{0\} \cup \{\frac{1}{n}; n \geq 1\} \cdot A$, the set of accumulation points of $\mathrm{Mld}_2^{tor}(A)$.

2 Toric log varieties

In this section we recall the definition of minimal log discrepancies and their explicit description in the toric case. The reader may consult [2] for more details.

A *log variety* (X, B) consists of a normal algebraic variety X, defined over an algebraically closed field k of characteristic zero, endowed with a finite combination $B =$

$\sum_i b_i B_i$ of Weil prime divisors B_i with non-negative real coefficients b_i, such that $K_X + B$ is $\mathbb{R}$-Cartier. Here K_X is the canonical divisor of X, computed as the Weil divisor of zeros and poles $(\omega)_X$ of a top rational form $\omega \in \Omega^{\dim(X)}_{k(X)/k}$; it is uniquely defined up to linear equivalence. The $\mathbb{R}$-Cartier property of $K_X + B$ means that locally on X, there exist finitely many non-zero rational functions $a_\alpha \in k(X)^\times$ and $r_\alpha \in \mathbb{R}$ such that $K_X + B = \sum_\alpha r_\alpha (a_\alpha)$.

Let $\mu\colon X' \to X$ be a proper birational morphism from a normal variety X' and let $E \subset X'$ be a prime divisor. Let ω be a top rational form on X, defining K_X, and let $K_{X'}$ be the canonical divisor defined by $\mu^*\omega$. The real number

$$a(E; X, B) = 1 + \mathrm{mult}_E(K_{X'} - \mu^*(K_X + B))$$

is called the *log discrepancy of* (X, B) *at* E. For a Grothendieck point $\eta \in X$, the *minimal log discrepancy of* (X, B) *at* η is defined as

$$a(\eta; X, B) = \inf_{\mu(E)=\bar{\eta}} a(E; X, B),$$

where the infimum is taken over all prime divisors E on proper birational maps $\mu\colon X' \to X$. This infimum is either $-\infty$, or a non-negative real number. In the latter case, (X, B) is said to have *log canonical singularities at* η and the invariant is computed as follows: By Hironaka, there exists a proper birational morphism $\mu\colon X' \to X$ such that X' is nonsingular, $\mu^{-1}(\bar{\eta})$ is a divisor on X', and there exists a simple normal crossings divisor $\sum_i E_i$ on X' which supports both $\mu^{-1}(\bar{\eta})$ and $K_{X'} - \mu^*(K_X + B)$. Then

$$a(\eta; X, B) = \min_{\mu(E_i)=\bar{\eta}} a(E_i; X, B).$$

Next we specialize these notions to the toric case. We employ standard terminology on toric varieties, cf. Oda [7]. A *toric log variety* is a log variety (X, B) such that X is a toric variety and B is torus invariant. Thus there exists a fan Δ in a lattice N such that $X = T_N \operatorname{emb}(\Delta)$ and $B = \sum_i b_i V(e_i)$, where $\{e_i\}_i$ is the set of primitive lattice points on the one-dimensional cones of Δ and $V(e_i) \subset X$ is the torus invariant prime Weil divisor corresponding to e_i. The canonical divisor is $K_X = \sum_i -V(e_i)$, and the $\mathbb{R}$-Cartier property of $K_X + B$ means that there exists a function $\psi\colon |\Delta| \to \mathbb{R}$ such that $\psi(e_i) = 1 - b_i$ for every i, and $\psi|_\sigma$ is linear for every cone $\sigma \in \Delta$. We may assume that (X, B) has log canonical singularities, which is equivalent to $\psi \geq 0$ or $b_i \in [0, 1]$ for all i.

Let $e \in N^{prim} \cap |\Delta|$ be a non-zero primitive vector. The barycentric subdivision with respect to e defines a subdivision $\Delta_e \prec \Delta$ and the exceptional locus of the birational morphism $T_N \operatorname{emb}(\Delta_e) \to T_N \operatorname{emb}(\Delta)$ is a prime divisor denoted E_e. It is easy to see that

$$a(E_e; X, B) = \psi(e).$$

Due to this property, ψ is called the *log discrepancy function of* (X, B).

Minimal log discrepancies of toric log varieties are computed as follows: Since these are local invariants, we only consider affine varieties and thus Δ consists of the faces of some strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$. We denote $X = T_N \operatorname{emb}(\sigma)$.

Assume first that $0 \in X$ is a torus invariant closed point (it is unique since X is affine). Using the existence of good resolutions in the toric category, it is easy to see that

$$a(0; X, B) = \min(\psi|_{N \cap \mathrm{relint}(\sigma)}).$$

For the general case, let $\eta \in X$ be a Grothendieck point. There exists a unique face $\tau \prec \sigma$ such that $\eta \in \mathrm{orb}(\tau)$. Let c and d be the codimension of $\mathrm{orb}(\tau)$ and η in X, respectively. The induced affine toric log variety

$$(X', B') = (T_{N \cap (\tau - \tau)} \,\mathrm{emb}(\tau), \sum_{e \in \tau(1)} \mathrm{mult}_{V(e)}(B) V(e))$$

has a unique torus invariant closed point $0'$, and we obtain

$$a(\eta; X, B) = \mathrm{mld}(0'; X', B') + d - c.$$

3 The set of toric minimal log discrepancies

Let $A \subseteq [0, 1]$ be a set containing 1.

Definition 3.1. For an integer $d \geq 1$, let $\mathrm{Mld}_d^{tor}(A)$ be the set of minimal log discrepancies $a(\eta; X, B)$, where $\eta \in X$ is a Grothendieck point of codimension d and (X, B) is a toric log variety whose minimal log discrepancies in codimension one belong to A.

It is easy to see that $\mathrm{Mld}_1^{tor}(A) = A$.

Definition 3.2. For an integer $d \geq 2$, define $V_d(A)$ to be the set of pairs $(x, a) \in (0, 1]^d \times A^d$ satisfying the following properties:
(i) $x \in \mathbb{Q}^d$.
(ii) $\mathrm{index}(x_i) | \mathrm{index}(x_1, \ldots, \hat{x}_i, \ldots, x_d)$ for $1 \leq i \leq d$.
(iii) $\sum_{i=1}^d (1 + (m-1)x_i - \lceil m x_i \rceil) a_i \geq 0$ for all $m \in \mathbb{Z}$.
For $x \in \mathbb{Q}^n$, $\mathrm{index}(x)$ denotes the smallest positive integer q such that $qx \in \mathbb{Z}^n$.

Note that property (ii) means that $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ are primitive vectors in the lattice $\mathbb{Z}^d + \mathbb{Z}x$. Also, it is enough to verify property (iii) for the finitely many integers $1 \leq m \leq \mathrm{index}(x) - 1$. For $(x, a) \in V_d(A)$ we denote

$$\langle x, a \rangle = \sum_{i=1}^d x_i a_i.$$

Proposition 3.3. *For $d \geq 2$, we have*

$$\mathrm{Mld}_d^{tor}(A) = \bigcup_{2 \leq s \leq d} \{\langle x, a \rangle; (x, a) \in V_s(A)\}.$$

Proof. (1) We first show that the right hand side is a subset of the left hand side. Fix $(x, a) \in V_s(A)$ for some $2 \leq s \leq d$.

If $s = d$, let $N = \mathbb{Z}^d + \mathbb{Z}x$ and let σ be the standard positive cone in $\mathbb{R}^d$, spanned by the standard basis $e_1, \ldots, e_d$ of $\mathbb{Z}^d$. Let $0 \in T_N \operatorname{emb}(\sigma)$ be the invariant closed point corresponding to σ. Then the affine toric log variety

$$(T_N \operatorname{emb}(\sigma), \sum_{i=1}^{d}(1-a_i)V(e_i))$$

has minimal log discrepancy at 0 equal to $\langle x, a\rangle$. Indeed, the log discrepancy function $\psi = \sum_{i=1}^d a_i e_i^*$ attains its minimum at x, and $\psi(x) = \langle x, a\rangle$. Therefore $\langle x, a\rangle \in \operatorname{Mld}_d^{tor}(A)$.

Assume now that $2 \leq s \leq d-1$. Let $e_1, \ldots, e_d$ be the standard basis of $\mathbb{Z}^d$, let $e_{d+1} = (d-s)e_1 + e_2 - \sum_{i=s+1}^d e_i$, let $v = \sum_{i=1}^s x_i e_i$ and let $N = \mathbb{Z}^d + \mathbb{Z}v$. Let σ be the cone in $\mathbb{R}^d$ generated by $e_1, \ldots, e_{d+1}$ and set $a_i = a_1$ for $s+1 \leq i \leq d$ and $a_{d+1} = a_2$. Then

$$0 \in (T_N \operatorname{emb}(\sigma), \sum_{i=1}^{d+1}(1-a_i)V(e_i))$$

is a d-dimensional germ of a toric log variety with minimal log discrepancy equal to $\langle x, a\rangle$.

Indeed, note first that $K_X + B$ is $\mathbb{R}$-Cartier since $a_2 = (d-s)a_1 + a_2 - \sum_{i=s+1}^d a_1$. Also, the log discrepancy function is $\psi = \sum_{i=1}^d a_i e_i^*$ and there exists $e = \sum_{i=1}^{d+1} y_i e_i \in N \cap \operatorname{relint}(\sigma)$ where the log discrepancy function ψ attains its minimum. We may assume $y_i \in [0,1]$ for every i. If $y_{d+1} \notin \mathbb{Z}$, then $y_{s+1} = \cdots = y_d = y_{d+1}$, hence $e = \sum_{i=1}^s y_i e_i$. Therefore $\psi(e) \geq \psi(v)$. If $y_{d+1} \in \mathbb{Z}$, then $\sum_{i=1}^s y_i e_i \in N \cap \operatorname{relint}(\sigma)$, hence $\psi(e) \geq \psi(\sum_{i=1}^s y_i e_i) \geq \psi(v)$. We conclude that ψ attains its minimum at v and therefore $\langle x, a\rangle = \psi(v) \in \operatorname{Mld}_d^{tor}(A)$.

(2) Let (X, B) be a toric log variety with codimension one log discrepancies in A and let $\eta \in X$ be a Grothendieck point of codimension d. We shall show that $a(\eta; X, B)$ belongs to the set on the right hand side.

There exists a unique cone σ in the fan defining X such that $\eta \in \operatorname{orb}(\sigma)$. Let c be the codimension of $\operatorname{orb}(\sigma)$ in X. Then $a(\eta; X, B)$ coincides with the minimal log discrepancy of the toric log variety

$$(T_{N\cap(\sigma-\sigma)} \operatorname{emb}(\sigma), \sum_{e\in\sigma(1)} \operatorname{mult}_{V(e)}(B)V(e)) \times \mathbb{A}_k^{d-c}.$$

in the invariant closed point 0. Therefore we may assume that X is affine and η is a torus invariant closed point 0.

We have $X = T_N \operatorname{emb}(\sigma)$, with $\dim \sigma = \dim N = d$, $B = \sum_{i\in I}(1-a_i)V(e_i)$ with $a_i \in A$ for every i. The log discrepancy function $\psi \in \sigma^\vee$ of (X, B) satisfies $\psi(e_i) = a_i$, and we have

$$\operatorname{mld}(0; X, B) = \min(\psi|_{N\cap\operatorname{relint}(\sigma)}).$$

There exists $e \in N \cap \operatorname{relint}(\sigma)$ such that $\operatorname{mld}(0; X, B) = \psi(e)$. By Carathéodory's Theorem (see [7], Theorem A.15), there exists a subset $\{1, \ldots, s\} \subseteq I$, with $2 \leq s \leq d$, such that $e_1, \ldots, e_s$ are linearly independent and e belongs to the relative interior of the cone spanned by $e_1, \ldots, e_s$. Let $e = \sum_{i=1}^s x_i e_i$, and denote $x = (x_1, \ldots, x_s) \in (0,1]^d, a = (a_1, \ldots, a_s) \in A^d$. It is clear that $\operatorname{mld}(0; X, B) = \langle x, a\rangle$, and we claim that $(x, a) \in V_s(A)$.

Indeed, it is clear that $x \in \mathbb{Q}^s$. Since e_i is a primitive lattice point of N, it is also primitive in the sublattice $\sum_{i=1}^s \mathbb{Z}e_i + \mathbb{Z}e$, which is equivalent to $\operatorname{index}(x_i) | \operatorname{index}(x_1, \dots, \hat{x}_i, \dots, x_s)$ for every $1 \leq i \leq s$. Finally, let $m \in \mathbb{Z}$. We have $\sum_{i=1}^s (1 + mx_i - \lceil mx_i \rceil)e_i \in N \cap \operatorname{relint}(\sigma)$, hence $\psi(\sum_{i=1}^s (1 + mx_i - \lceil mx_i \rceil)e_i) \geq \psi(e)$. Equivalently, $\sum_{i=1}^s (1 + (m-1)x_i - \lceil mx_i \rceil)a_i \geq 0$ and therefore $(x, a) \in V_s(A)$. □

4 The set $\tilde{V}_d(A)$

By Proposition 3.3, the limiting behaviour of toric minimal log discrepancies is controlled by the limiting behaviour of the sets $V_d(A)$. The rationality properties (i) and (ii) defining $V_d(A)$ do not behave well with respect to limits, and for this reason we enlarge $V_d(A)$ to a new set $\tilde{V}_d(A)$, defined only by property (iii), which turns out to have good inductive properties and limiting behaviour.

Definition 4.1. Let $A \subseteq [0, 1]$ be a subset containing 1. Define

$$\tilde{V}_d(A) = \{(x, a) \in (0, 1]^d \times A^d; \sum_{i=1}^d (1 + (m-1)x_i - \lceil mx_i \rceil)a_i \geq 0, \forall m \in \mathbb{Z}\}.$$

Equivalently, $\tilde{V}_d(A)$ is the set of pairs $(x, a) \in (0, 1]^d \times A^d$ such that the group $\mathbb{Z}^d + \mathbb{Z}x$ does not intersect the set $\{y \in (0, 1]^d; \langle y - x, a \rangle < 0\}$. As before, we denote $\langle x, a \rangle = \sum_{i=1}^d x_i a_i$.

Lemma 4.2. *The following equality holds*

$$\tilde{V}_1(A) = ((0, 1] \times \{0\}) \cup (\{\frac{1}{n}; n \geq 1\} \times A),$$

where the first term is missing if $0 \notin A$. *In particular,*

$$\{\langle x, a \rangle; (x, a) \in \tilde{V}_1(A)\} = \bigcup_{n=1}^{\infty} \frac{1}{n} \cdot A.$$

Proof. Let $x \in (0, 1]$ such that $1 + (m-1)x - \lceil mx \rceil \geq 0$ for every integer m. Equivalently, we have

$$\sup_{m \in \mathbb{Z}} (\lceil mx \rceil - mx) \leq 1 - x.$$

Assume by contradiction that $x \notin \mathbb{Q}$. Then the set $\{\lceil mx \rceil - mx\}_{m \geq 1}$ is dense in $[0, 1]$ (cf. [5], Chapter IV), hence $\sup_{m \in \mathbb{Z}} (\lceil mx \rceil - mx) = 1$. We obtain $1 \leq 1 - x$, hence $x = 0$, a contradiction.

Therefore $x = \frac{p}{q}$, for integers $1 \leq p \leq q$ with $\gcd(p, q) = 1$. The above inequality becomes

$$1 - \frac{1}{q} = \max_{m \in \mathbb{Z}} (\lceil mx \rceil - mx) \leq 1 - \frac{p}{q},$$

hence $p = 1$. Therefore $x = \frac{1}{q}$. □

We will need the following result of Lawrence. Note that property (ii) is a consequence of (i).

Theorem 4.3 ([6]). *Let $T = \mathbb{R}^d/\mathbb{Z}^d$ be a real torus.*

(i) Let $U \subset T$ be an open subset. The set of closed subgroups of T which do not intersect U has only finitely many maximal elements with respect to inclusion.

(ii) The set of finite unions of closed subgroups of T satisfies the descending chain condition.

Theorem 4.4. *Assume that A satisfies the ascending chain condition. Then the set $\{\langle x, a\rangle; (x, a) \in \tilde{V}_d(A)\}$ satisfies the ascending chain condition.*

Proof. Assume first that $d = 1$. By Lemma 4.2,

$$\{\langle x, a\rangle; (x, a) \in \tilde{V}_1(A)\} = \{\frac{1}{n}; n \geq 1\} \cdot A.$$

Both sets $\{\frac{1}{n}; n \geq 1\}$ and A are nonnegative and satisfy the ascending chain condition, hence their product satisfies the ascending chain condition.

Now suppose $d \geq 2$ and assume by induction the result holds for smaller values of d. Assume by contradiction that $\{(x^n, a^n)\}_{n\geq 1}$ is a sequence in $\tilde{V}_d(A)$ such that

$$\langle x^n, a^n\rangle < \langle x^{n+1}, a^{n+1}\rangle \text{ for } n \geq 1.$$

Since A satisfies the ascending chain condition, we may assume after passing to a subsequence that

$$a_i^n \geq a_i^{n+1}, \forall n \geq 1, \forall 1 \leq i \leq d.$$

Assume first that $x^n \notin (0, 1)^d$ for infinitely many n's. After passing to a subsequence, we may assume $x_1^n = 1$ for every n. Write $x^n = (1, \bar{x}^n)$ and $a^n = (a_1^n, \bar{a}^n)$. Then $\langle \bar{x}^n, \bar{a}^n\rangle < \langle \bar{x}^{n+1}, \bar{a}^{n+1}\rangle$ for every $n \geq 1$, which contradicts the ACC property of the set $\{\langle \bar{x}, \bar{a}\rangle; (\bar{x}, \bar{a}) \in \tilde{V}_{d-1}(A)\}$.

Assume now that $x^n \in (0, 1)^d$ for every n. We set

$$U^n = \{x \in (0, 1)^d; \langle x - x^n, a^n\rangle < 0\}$$

and regard U^n as an open subset of the torus $T^d = \mathbb{R}^d/\mathbb{Z}^d$. Let X^n be the union of the subgroups of T^d which do not intersect U^n. By Theorem 4.3.(i), X^n is a finite union of closed subgroups of T^d. It is easy to see that $U^n \subseteq U^{n+1}$, hence $X^n \supseteq X^{n+1}$ for $n \geq 1$.

Since $(x^n, a^n) \in \tilde{V}_d(A)$, we have $U^n \cap (\mathbb{Z}^d + \mathbb{Z}x^n) = \emptyset$. Therefore $x^n \in X^n$ for every n. We have

$$\langle x^n, a^{n+1}\rangle \leq \langle x^n, a^n\rangle < \langle x^{n+1}, a^{n+1}\rangle.$$

Then $x^n \in U^{n+1}$, hence $x^n \notin X^{n+1}$. Therefore $X^n \supsetneqq X^{n+1}$ for every $n \geq 1$, contradicting Theorem 4.3.(ii). ∎

Lemma 4.5. *The following properties hold:*

(1) If A is a closed set, then $\tilde{V}_d(A)$ is a closed subset of $(0,1]^d \times A^d$.

(2) Identify $(0,1]^s$ with the face $x_{s+1} = \cdots = x_d = 1$ of $(0,1]^d$. Then

$$\tilde{V}_d(A) \cap (0,1]^s = \tilde{V}_s(A).$$

(3) Identify $[0,1]^s$ with the face $x_{s+1} = \cdots = x_d = 0$ of $[0,1]^d$ and assume that A is a closed set. Then

$$\overline{\tilde{V}_d(A)} \cap (0,1]^s = \tilde{V}_s(A).$$

Proof. (1) Let $(x,a) \in (0,1]^d \times A^d$ such that there exists a sequence $\{(x^n, a^n)\}_{n\geq 1}$ in $\tilde{V}_d(A)$ with $x = \lim_{n\to\infty} x^n$ and $a = \lim_{n\to\infty} a^n$. Fix $m \in \mathbb{Z}$. By assumption, we have

$$\sum_{i=1}^d (1 + (m-1)x_i^n - \lceil m x_i^n \rceil) a_i^n \geq 0, \forall n \geq 1.$$

There exists a positive integer $n(m)$ such that $\lceil m x_i^n \rceil \geq \lceil m x_i \rceil$ for every $1 \leq i \leq d$ and every $n \geq n(m)$. Therefore

$$\sum_{i=1}^d (1 + (m-1)x_i^n - \lceil m x_i \rceil) a_i^n \geq 0, \forall n \geq n(m),$$

Letting n tend to infinity, we obtain

$$\sum_{i=1}^d (1 + (m-1)x_i - \lceil m x_i \rceil) a_i \geq 0.$$

Since m was arbitrary, we conclude that $(x,a) \in \tilde{V}_d(A)$.

(2) This is clear.

(3) Assume that we have a sequence $\{(x^n, a^n)\}_{n\geq 1} \subset \tilde{V}_d(A)$ such that $\lim_{n\to\infty} x^n = (x, 0, \ldots, 0) \in (0,1]^s$ and $\lim_{n\to\infty} a^n = (a, a_{s+1}, \ldots, a_d)$. Let m be a positive integer. Note that for $s+1 \leq i \leq d$ we have $m x_i^n \in (0,1]$ for $n \geq n(m)$, hence

$$\lim_{n\to\infty} (1 + (m-1)x_i^n - \lceil m x_i^n \rceil) = 0 \text{ for } s+1 \leq i \leq d.$$

Therefore $\sum_{i=1}^s (1 + (m-1)x_i - \lceil m x_i \rceil) a_i \geq 0$ for every $m \geq 1$. Since $\mathbb{Z}^s + \mathbb{Z}x$ is included in the closure of $\mathbb{Z}^d + \mathbb{Z}_{\geq 0} x$, we obtain $\sum_{i=1}^s (1 + (m-1)x_i - \lceil m x_i \rceil) a_i \geq 0$ for $m \leq -1$ as well. Therefore $(x,a) \in \tilde{V}_s(A)$, proving the direct inclusion.

For the converse, just note that $(x,a) \in \tilde{V}_s(A)$ is the limit of the sequence $((x, \frac{1}{n}, \ldots, \frac{1}{n}), (a, 1, \ldots, 1)) \in \tilde{V}_d(A)$. □

Definition 4.6. For $x \in \mathbb{R}$ and $m \in \mathbb{Z}$, define

$$x^{(m)} = 1 + mx - \lceil mx \rceil.$$

Note that this operation induces a selfmap of the half-open interval $(0,1]$. For $x \in \mathbb{R}^d$ and $m \in \mathbb{Z}$, define $x^{(m)} \in \mathbb{R}^d$ componentwise.

Since $(0,1]^d \cap (\mathbb{Z}^d + \mathbb{Z}x) = \{x^{(m)}; m \in \mathbb{Z}\}$, we have the equivalent description

$$\tilde{V}_d(A) = \{(x,a) \in (0,1]^d \times A^d; \langle x^{(m)} - x, a\rangle \geq 0, \forall m \in \mathbb{Z}\}.$$

Lemma 4.7. *Let $x \in (0,1]^d$ and let $a \in A^d$ such that $a_i > 0$ for $1 \leq i \leq d$. Then there exists a relatively open neighborhood $x \in U \subseteq (0,1]^d$ such that if $y \in U$ and $\langle y^{(m)} - x, a\rangle \geq 0$ for every $m \in \mathbb{Z}$, then $\langle y - x, a\rangle = 0$.*

Proof. (1) Assume first that $x \in (0,1)^d$. By Theorem 4.3.(ii), the set of closed subgroups of $\mathbb{R}^d$ which contain $\mathbb{Z}^d$ and do not intersect the nonempty open set $\{y \in (0,1)^d; \langle y - x, a\rangle < 0\}$ has finitely many maximal elements with respect to inclusion, say $H_1, \ldots, H_l$.

If $x \in H_1$, then H_1 is a rational affine subspace of $\mathbb{R}^d$ in an open neighborhood $x \in U_1 \subset (0,1)^d$. Let $v \in H_1 - x$. Since $x \in (0,1)^d$, there exists $\epsilon > 0$ such that $x + tv \in H_1 \cap (0,1)^d$ for $|t| < \epsilon$. In particular, $\langle x + tv - x, a\rangle \geq 0$, that is $t\langle v, a\rangle \geq 0$ for $|t| < \epsilon$. We infer that $\langle v, a\rangle = 0$. Therefore $H_1 \cap U_1$ is contained in $\{y \in (0,1)^d; \langle y - x, a\rangle = 0\}$. If $x \notin H_1$, then $U_1 = (0,1)^d \setminus H_1$ is an open neighborhood of x.

Repeating this procedure, we obtain a neighborhood U_i of x, for each closed subgroup H_i. The intersection $U = U_1 \cap \cdots \cap U_l$ is the desired neighborhood.

(2) We may assume after a reordering that $x_i = 1$ for $1 \leq i \leq s$ and $x_i \in (0,1)$ for $s < i \leq n$. If $s = n$, we may take $U = (0,1]^d$. Assume now that $s < n$. By [5], Chapter IV, there exists a *negative* integer m_0 such that

$$\langle x^{(m_0)} - x, a\rangle < \min_{i=1}^{s} a_i.$$

Let $y \in (0,1]^s \times \prod_{i=s+1}^d (\frac{\lceil m_0 x_i\rceil}{m_0}, \frac{\lceil m_0 x_i\rceil - 1}{m_0})$ such that $\langle y^{(m)} - x, a\rangle \geq 0$ for every $m \in \mathbb{Z}$. We claim that $y_1 = \cdots = y_s = 1$. Indeed, assume by contradiction that $y_j < 1$ for some $1 \leq j \leq s$. A straightforward computation gives

$$\langle y^{(m_0)} - x, a\rangle - m_0\langle y - x, a\rangle = \langle x^{(m_0)} - x, a\rangle + \sum_{i=1}^{d} (\lceil m_0 x_i\rceil - \lceil m_0 y_i\rceil) a_i.$$

By the choice of y, we obtain

$$\sum_{i=1}^{d} (\lceil m_0 x_i\rceil - \lceil m_0 y_i\rceil) a_i = \sum_{i=1}^{s} (m_0 - \lceil m_0 y_i\rceil) a_i \leq -a_j,$$

hence $0 \leq \langle x^{(m_0)} - x, a\rangle - a_j$. This contradicts our choice of m_0.

Let $\bar{x} = (x_{s+1}, \ldots, x_d), \bar{y} = (y_{s+1}, \ldots, y_d), \bar{a} = (a_{s+1}, \ldots, a_d)$. We have $(\bar{x}, \bar{a}) \in \tilde{V}_{d-s}(A)$ and $\langle \bar{y}^{(m)} - \bar{x}, \bar{a}\rangle \geq 0$ for every $m \in \mathbb{Z}$. From Step 1, there exists an open neighborhood $\bar{x} \in \bar{U} \subset (0,1)^s$ such that if $\bar{y} \in \bar{U}$ then $\langle \bar{y} - \bar{x}, \bar{a}\rangle = 0$. Then

$$U = (0,1]^s \times (\bar{U} \cap \prod_{i=s+1}^{d} (\frac{\lceil m_0 x_i\rceil}{m_0}, \frac{\lceil m_0 x_i\rceil - 1}{m_0})).$$

satisfies the required properties. $\square$

Lemma 4.8. *The following equality holds for $d \geq 1$ and $a \in A^d$:*

$$\{\langle x, a\rangle; (x, a) \in \tilde{V}_d(A), x \in \mathbb{Q}^d\} = \{\langle x, a\rangle; (x, a) \in \tilde{V}_d(A)\}.$$

Proof. Let $(x, a) \in \tilde{V}_d(A)$. We have $x_1, \ldots, x_s < 1$ and $x_{s+1} = \ldots = x_d = 1$, where $0 \leq s \leq d$. If $s = 0$, then $x \in \mathbb{Q}^d$ and we are done. Assume $s \geq 1$ and set $\bar{x} = (x_1, \ldots, x_s)$ and $\bar{a} = (a_1, \ldots, a_s)$. Then $(\bar{x}, \bar{a}) \in \tilde{V}_s(A)$. Since $\bar{x} \in (0, 1)^s$, there exists a closed subgroup $\mathbb{Z}^s \subseteq \bar{H} \subseteq \mathbb{R}^s$ such that $\bar{x} \in \bar{H} \cap U_{\bar{x}} \subset \{\bar{z}; \langle \bar{z} - \bar{x}, \bar{a}\rangle = 0\}$, by Step 1 of the proof of Lemma 4.7. Since $\bar{H}$ is rational, there exists $\bar{z} \in \mathbb{Q}^s \cap H \cap U_{\bar{x}}$. Set $x' = (\bar{z}, 1, \ldots, 1)$. Then $(x', a) \in \tilde{V}_d(A)$, $\langle x, a\rangle = \langle x', a\rangle$ and $x' \in \mathbb{Q}^d$. □

Proposition 4.9. *Assume that A has no positive accumulation points. Then the set of accumulation points of $\{\langle x, a\rangle; (x, a) \in \tilde{V}_d(A)\}$ is*

$$\{0\} \cup \bigcup_{1 \leq d' \leq d-1} \{\langle x, a\rangle; (x, a) \in \tilde{V}_{d'}(A)\}.$$

Proof. Let $r > 0$ be an accumulation point, that is there exists a sequence $(x^n, a^n) \in \tilde{V}_d(A)$ with $r = \lim_{n\to\infty}\langle x^n, a^n\rangle$ and $r \neq \langle x^n, a^n\rangle$ for every $n \geq 1$. By compactness, we may assume after passing to a subsequence that $\lim_{n\to\infty} x^n = x \in [0, 1]^d$ and $\lim_{n\to\infty} a^n = a \in [0, 1]^d$ exist. We have $r = \langle x, a\rangle$.

We claim that $a_i x_i = 0$ for some i. Indeed, assume by contradiction that $a_i x_i > 0$ for every $1 \leq i \leq d$. Since A has no nonzero accumulation points, we obtain $a^n = a$ for $n \geq 1$. Let $U_x \subset (0, 1]^d$ be the relative neighborhood of x associated to (x, a) in Lemma 4.7. Then $x^n \in U_x$ for $n \geq n_0$. If $\langle x^n - x, a\rangle \geq 0$, then $(x^n, a) \in \tilde{V}_d(A)$ implies that $\langle z - x, a\rangle \geq 0$ for every $z \in (\mathbb{Z}^d + \mathbb{Z}x^n) \cap (0, 1]^d$. Therefore $\langle x^n - x, a\rangle = 0$. This means $\langle x^n, a\rangle = r$, a contradiction.

Therefore $\langle x^n, a\rangle < r$ for every n. Since A has no positive accumulation points, it satisfies the ascending chain condition. Therefore the sequence $(\langle x^n, a\rangle)_{n\geq 1}$ satisfies the ascending chain condition as well, by Theorem 4.4. This is a contradiction.

We may assume $a_i x_i > 0$ for $1 \leq i \leq d'$ and $a_i x_i = 0$ for $d' + 1 \leq i \leq d$. We have $d' \geq 1$, since $\langle x, a\rangle > 0$. Denote $\bar{x} = (x_1, \ldots, x_{d'})$ and $\bar{a} = (a_1, \ldots, a_{d'})$. We have $r = \langle \bar{x}, \bar{a}\rangle$ and $(\bar{x}, \bar{a}) \in \tilde{V}_{d'}(A)$ by Theorem 4.5.

For the converse, note that

$$((\frac{1}{k}, \ldots, \frac{1}{k}), (1, \ldots, 1)) \in \tilde{V}_d(A)$$

and $\langle(\frac{1}{k}, \ldots, \frac{1}{k}), (1, \ldots, 1)\rangle = \frac{d}{k}$ accumulates to 0. Let now $(x, a) \in \tilde{V}_{d'}(A)$ for $1 \leq d' \leq d - 1$. Define $x_k = (x', \frac{1}{k}, \ldots, \frac{1}{k})$ and $a = (a', 1, \ldots, 1)$. Then $(x_k, a) \in \tilde{V}_d(A)$ and $\langle x_k, a\rangle = \langle x', a'\rangle + \frac{d-d'}{k}$ accumulates to $\langle x', a'\rangle$. □

Remark 4.10. Proposition 4.9 is false if A has a positive accumulation point. For example, let $a > 0$ be an accumulation point of a sequence of elements $a_k \in A$. Then $((1, \ldots, 1), (a_k, 1, \ldots, 1)) \in V_d(A)$ and $\langle(1, \ldots, 1), (a_k, 1, \ldots, 1)\rangle$ accumulates to $d - 1 + a$, which clearly does not correspond to any element of $\tilde{V}_{d'}(A)$, for $d' \leq d - 1$.

The set $\tilde{V}_d(A)$ is strictly larger that $V_d(A)$. For example, $(\frac{1}{2}, 1)$ and $(\frac{l-1}{2l}, \frac{1}{l})$ $(l \geq 2)$ are rational points of $\tilde{V}_d(\{1\}) \setminus V_d(\{1\})$. However, the following property holds.

Lemma 4.11. *The following inclusion holds:*

$$\{\langle x, a\rangle; (x, a) \in \tilde{V}_d(A)\} \subseteq \{\langle x, a\rangle; (x, a) \in V_d(\{\frac{1}{n}; n \geq 1\} \cdot A)\}.$$

Proof. Let $r = \langle x, a\rangle$ for some $(x, a) \in \tilde{V}_d(A)$. By Lemma 4.8, we may assume that $x \in \mathbb{Q}^d$. We may assume $a_i > 0$ for every i. Let $e_1, \ldots, e_d$ be the standard basis of $\mathbb{R}^d$, spanning the standard cone σ, let $e = \sum_{i=1}^d x_i e_i$ and let $N = \sum_{i=1}^d \mathbb{Z}e_i + \mathbb{Z}e$. If we set $\psi = \sum_{i=1}^d a_i e_i^*$, then we have

$$\min(\psi|_{N \cap \mathrm{relint}(\sigma)}) = \psi(e) = r.$$

There exists positive integers $n_i \geq 1$ such that $e_i' = \frac{1}{n_i} e_i$ are primitive elements of the lattice N. In the new coordinates, we have $\psi = \sum_{i=1}^d \frac{a_i}{n_i} e_i'^*$ and $e = \sum_{i=1} n_i x_i e_i'$. Since ψ attains its minimum at e and all a_i's are positive, we infer that $n_i x_i < 1$ for every i. Set $a_i' = \frac{a_i}{n_i}$ and $x_i' = n_i x_i$. Then $(x', a') \in V_d(\{\frac{1}{n}; n \geq 1\} \cdot A)$ and $\langle x', a'\rangle = r$. □

Corollary 4.12. *Assume that* $A = \{\frac{1}{n}; n \geq 1\} \cdot A$. *Then*

$$\{\langle x, a\rangle; (x, a) \in V_d(A)\} = \{\langle x, a\rangle; (x, a) \in \tilde{V}_d(A)\}.$$

5 Accumulation points of $\mathrm{Mld}_d^{tor}(A)$

Theorem 5.1. *The following properties hold:*

(1) *If A satisfies the ascending chain condition, then so does* $\mathrm{Mld}_d^{tor}(A)$.

(2) *Assume that A has no positive accumulation points. Then the set of accumulation points of* $\mathrm{Mld}_d^{tor}(A)$ *is included in*

$$\{0\} \cup \bigcup_{1 \leq d' \leq d-1} \mathrm{Mld}_{d'}^{tor}(\{\frac{1}{n}; n \geq 1\} \cdot A).$$

The inclusion is an equality if $\{\frac{1}{n}; n \geq 1\} \cdot A \subset A$.

(3) *Assume that A has no positive accumulation points and* $\{\frac{1}{n}; n \geq 1\} \cdot A \subset A$. *Then* $\mathrm{Mld}_d^{tor}(A)$ *is a closed set if and only if* $0 \in A$.

Proof. The inclusion $V_d(A) \subset \tilde{V}_d(A)$ and Proposition 3.3 give

$$\mathrm{Mld}_d^{tor}(A) \subseteq \bigcup_{2 \leq d' \leq d} \{\langle x, a\rangle; (x, a) \in \tilde{V}_{d'}(A)\}.$$

(1) The set $\mathrm{Mld}_d^{tor}(A)$ is a subset of a finite union of sets satisfying the ascending chain condition, by Theorem 4.4. Therefore $\mathrm{Mld}_d^{tor}(A)$ satisfies the ascending chain condition.

(2) Assume that A has no nonzero accumulation points. By Proposition 4.9 and Lemma 4.11, the accumulation points of $\mathrm{Mld}^{tor}_d(A)$ belong to the set

$$\{0\} \cup \bigcup_{1 \le d' \le d-1} \mathrm{Mld}^{tor}_{d'}(\{\frac{1}{n}; n \ge 1\} \cdot A).$$

Assuming moreover that $\{\frac{1}{n}; n \ge 1\} \cdot A \subseteq A$, we will show that all points of the above set are accumulation points of $\mathrm{Mld}^{tor}_d(A)$. If $(x,a) \in V_{d'}(A)$, then

$$((x,1,\ldots,1),(a,\frac{1}{n},\ldots,\frac{1}{n})) \in V_d(A),$$

and $\langle (x,1,\ldots,1),(a,\frac{1}{n},\ldots,\frac{1}{n})\rangle = \langle x,a\rangle + \frac{d-d'}{n}$ accumulates to $\langle x,a\rangle$. Similarly,

$$((1,1,\ldots,1),(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n})) \in V_d(A)$$

and $\langle (1,1,\ldots,1),(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n})\rangle = \frac{d}{n}$ accumulates to 0. This proves the claim.

(3) Assume that $\mathrm{Mld}^{tor}_d(A)$ is a closed set. Since

$$((\frac{1}{k},\ldots,\frac{1}{k}),(a,\ldots,a)) \in V_d(A),$$

we infer that $0 = \lim_{k\to\infty} \frac{da}{k} \in \mathrm{Mld}^{tor}_d(A)$, which implies $0 \in A$.

Conversely, assume $0 \in A$. If $(x,a) \in V_{d'}(A)$ then

$$((x,1,\ldots,1),(a,0,\ldots,0)) \in V_d(A)$$

and $\langle (x,1,\ldots,1),(a,0,\ldots,0)\rangle = \langle x,a\rangle$. We infer from (3) that $\mathrm{Mld}^{tor}_d(A)$ is a closed set. $\square$

Lemma 5.2. *Assume that A has no positive accumulation points. Then the following properties hold:*

(1) The set of accumulation points of $\mathrm{Mld}^{tor}_2(A)$ is $\{0\} \cup \bigcup_{k\ge 1} \frac{1}{k}A$.

(2) The set $\mathrm{Mld}^{tor}_2(A)$ is closed if and only if $0 \in A$.

Proof. (1) From Theorem 5.1, all accumulation points are of this form. Conversely, fix $a \in A$ and $k \in \mathbb{Z}_{\ge 1}$. Then $((\frac{1}{kn+1},\frac{n}{nk+1}),(a,a)) \in V_2(A)$ is a sequence converging to $((0,\frac{1}{k}),(a,a))$, hence $\frac{a}{k}$ is an accumulation point of $\mathrm{Mld}^{tor}_2(A)$. Since k is arbitrary, we infer that 0 is an accumulation point as well.

(2) Assume that $\mathrm{Mld}^{tor}_2(A)$ is a closed set. Then $0 \in \mathrm{Mld}^{tor}_2(A)$, which implies $0 \in A$.

Assume now that $0 \in A$. Then for $a \in A$ and $k \in \mathbb{Z}_{\ge 1}$ we have

$$\frac{a}{k} = \langle (\frac{1}{k},\frac{1}{k}),(0,a)\rangle \in \mathrm{Mld}^{tor}_2(A).$$

From (1), these are all possible accumulation points, hence $\mathrm{Mld}^{tor}_2(A)$ is a closed set. $\square$

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