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Nef dimension of minimal models

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Abstract. We reduce the Abundance Conjecture in dimension 4 to the following numerical statement: if the canonical divisor K is nef and has maximal nef dimension, then K is big. From this point of view, we classify in dimension 2 nef divisors which have maximal nef dimension, but which are not big.

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0. Introduction

A *minimal model* is a complex projective variety X with at most canonical singularities, whose canonical divisor K is numerically effective (nef): $K \cdot C \geq 0$ for every curve $C \subset X$. Up to dimension three, minimal models have a geometrical characterization (Kawamata [9,12], Miyaoka [14–16]) :

Abundance Conjecture. [11] Let X be a minimal model. Then the linear system $|kK|$ is base point free, for some positive integer k.

In dimension four, it is enough to show that X has positive Kodaira dimension if K is not numerically trivial (Kawamata [9], Mori [17]).

A direct approach is to first construct the morphism associated to the expected base point free pluricanonical linear systems:

$$f\colon X \to \operatorname{Proj}(\oplus_{k\geq 0} H^0(X, kK)).$$

Since K is nef, f can be characterized numerically: it is the unique morphism with connected fibers which contracts exactly the curves $C \subset X$ with $K \cdot C = 0$. Tsuji [24] and Bauer et al [2] have recently solved this existence problem *birationally*: for any nef divisor D on X, there exists a rational dominant map $f\colon X \dashrightarrow Y$ such that f is regular over the generic point of Y and a very general curve C is

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contracted by f if and only if $D \cdot C = 0$. This rational map is called the *nef reduction* of D, and $n(X, D) := \dim(Y)$ is called the *nef dimension* of D. The nef reduction map is non-trivial, except for the two extremal cases:

(i) $n(X, K) = 0$: K is numerically trivial in this case [2], and Abundance is known (Kawamata [10]).
(ii) $n(X, K) = \dim(X)$: $K \cdot C > 0$ for very general curves $C \subset X$. The nef reduction map is birational, and we say that K has maximal nef dimension.

Our main result is that Abundance holds for a minimal model X if the nef reduction map is non-trivial and the Log Minimal Model Program and Log Abundance hold in dimension $n(X, K)$. The latter two conjectures are known to hold up to dimension three (Shokurov [21], Keel, Matsuki, McKernan [13]), hence we obtain

Theorem 0.1. *Let X be a minimal model with $n(X, K) \leq 3$. Then the linear system $|kK|$ is base point free for some positive integer k.*

The Base Point Free Theorem (Kawamata, Shokurov [11]) states that Abundance holds if the canonical class K is big. As a corollary of Theorem 0.1, we obtain that the 4-dimensional case of the Abundance Conjecture is equivalent to the following

Conjecture 0.2. *Let X be a minimal 4-fold. If $K \cdot C > 0$ for very general curves $C \subset X$, then K is big.*

We stress that this statement is *numerical*: since K is nef, K is big if and only if $K^{\dim(X)} > 0$. For this reason, it is important to investigate how far are (adjoint) divisors of maximal nef dimension from being big. Questions of similar type have appeared in the literature: a divisor D is *strictly nef* (Serrano [20]) if $D \cdot C > 0$ for every curve $C \subset X$. Up to dimension 3, it is known that $\pm K$ is strictly nef if and only if $\pm K$ is ample (see [20,25] and the references there). We point out that Conjecture 0.2 is false for the anti-canonical divisor $-K$ (which, at least in dimension two, is the only exception below):

Theorem 0.3. *Let X be a smooth projective surface. Assume that D is a nef Cartier divisor of maximal nef dimension, which is not big. Then exactly one of the following cases occurs:*

(1) The divisor $K + tD$ is big for $t > 2$.
(2) There exists a birational contraction $f: X \to Y$ and there exists $t \in (0, 2]$ such that $D = f^*(D_Y)$ and $K_Y + tD_Y \equiv 0$. Moreover, D is effective up to algebraic equivalence. In Sakai's classification table [18], Y is either a degenerate Del Pezzo, or an elliptic ruled surface of type II_c, II_c^*.

Theorem 0.1 is proved in several steps. The properties of the nef reduction map f and the numerically trivial case of Abundance [10] imply that f is birational to a parabolic fiber space $f' \colon X' \to Y'$, and the canonical class K descends to a divisor P on Y'. After an idea of Fujita [6], it is enough to show that P is the semi-positive part in the Fujita decomposition associated to a log variety (Y', Δ): the semi-ampleness of P follows then from the Log Minimal Model Program and Log Abundance applied to (Y', Δ). The key ingredient in this argument is an adjunction formula for the parabolic fiber space f' (Kawamata [8, 10], Fujino, Mori [4, 10]), similar to Kodaira's formula for elliptic surfaces. We expect that the logarithmic version of Theorem 0.1 follows from the same argument, provided that Kawamata's adjunction formula [8] is extended to the logarithmic case (see also Fukuda [7]).

Finally, Theorem 0.3 follows from the classification of surfaces and generalizes a result of Serrano [20].

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1. Preliminary

A *variety* is a reduced and irreducible separable scheme of finite type, defined over an algebraically closed field of characteristic zero. A *contraction* is a proper morphism $f \colon X \to Y$ such that $\mathcal{O}_Y = f_*\mathcal{O}_X$.

1-A. Divisors

Let X a normal variety, and let $L \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$. An L-Weil divisor is an element of $Z^1(X) \otimes_{\mathbb{Z}} L$. Two $\mathbb{R}$-Weil divisors D_1, D_2 are *L-linearly equivalent*, denoted $D_1 \sim_L D_2$, if there exist $q_i \in L$ and rational functions $\varphi_i \in k(X)^\times$ such that $D_1 - D_2 = \sum_i q_i(\varphi_i)$. An $\mathbb{R}$-Weil divisor D is called

(i) *L-Cartier* if $D \sim_L 0$ in a neighborhood of each point of X.
(ii) *nef* if D is $\mathbb{R}$-Cartier and $D \cdot C \geq 0$ for every curve $C \subset X$.
(iii) *ample* if X is projective and the numerical class of D belongs to the real cone generated by the numerical classes of ample Cartier divisors.
(iv) *semi-ample* if there exists a contraction $\Phi \colon X \to Y$ and an ample $\mathbb{R}$-divisor H on Y such that $D \sim_{\mathbb{R}} \Phi^* H$. If D is rational, this is equivalent to the linear system $|kD|$ being base point free for some k.
(v) *big* if there exists $C > 0$ such that $\dim H^0(X, kD) \geq Ck^{\dim(X)}$ for k sufficiently large and divisible. By definition,

$$H^0(X, kD) = \{a \in k(X)^\times; (a) + kD \geq 0\} \cup \{0\}.$$

The *Iitaka dimension* of D is $\kappa(X, D) = \max_{k \geq 1} \dim \Phi_{|kD|}(X)$, where $\Phi_{|kD|}: X \dashrightarrow \mathbb{P}(|kD|)$ is the rational map associated to the linear system $|kD|$. If all the linear systems $|kD|$ are empty, $\kappa(X, D) = -\infty$. If D is nef, the *numerical dimension* $\nu(X, D)$ is the largest non-negative integer k such that there exists a k-dimensional cycle $C \subset X$ with $D^k \cdot C \neq 0$.

1-B. B-divisors (V.V. Shokurov [22,23])

An *L-b-divisor* $\mathbf{D}$ of X is a family $\{\mathbf{D}_{X'}\}_{X'}$ of L-Weil divisors indexed by all birational models of X, such that $\mu_*(\mathbf{D}_{X''}) = \mathbf{D}_{X'}$ if $\mu: X'' \to X'$ is a birational contraction.

Equivalently, $\mathbf{D} = \sum_E \mathrm{mult}_E(\mathbf{D})E$ is a L-valued function on the set of all geometric valuations of the field of rational functions $k(X)$, having finite support on some (hence any) birational model of X.

Example 1. (1) Let ω be a top rational differential form of X. The associated family of divisors $\mathbf{K} = \{(\omega)_{X'}\}_{X'}$ is called the *canonical b-divisor* of X.

(2) A rational function $\varphi \in k(X)^\times$ defines a b-divisor $\overline{(\varphi)} = \{(\varphi)_{X'}\}_{X'}$.

(3) An $\mathbb{R}$-Cartier divisor D on a birational model X' of X defines an $\mathbb{R}$-b-divisor $\overline{D}$ such that $(\overline{D})_{X''} = \mu^* D$ for every birational contraction $\mu: X'' \to X'$.

An $\mathbb{R}$-b-divisor $\mathbf{D}$ is called *L-b-Cartier* if there exists a birational model X' of X such that $\mathbf{D}_{X'}$ is L-Cartier and $\mathbf{D} = \overline{\mathbf{D}_{X'}}$. In this case, we say that $\mathbf{D}$ *descends to* X'. An $\mathbb{R}$-b-divisor $\mathbf{D}$ is *b-nef* (*b-semi-ample*, *b-big*, *b-nef and good*) if there exists a birational contraction $X' \to X$ such that $\mathbf{D} = \overline{\mathbf{D}_{X'}}$ and $\mathbf{D}_{X'}$ is nef (semi-ample, big, nef and good).

1-C. Log pairs

A *log pair* (X, B) is a normal variety X endowed with a $\mathbb{Q}$-Weil divisor B such that $K + B$ is $\mathbb{Q}$-Cartier. A *log variety* is a log pair (X, B) such that B is effective. The *discrepancy* $\mathbb{Q}$*-b-divisor* of a log pair (X, B) is the $\mathbb{Q}$-b-divisor of X defined by

$$\mathbf{A}(X, B) = \mathbf{K} - \overline{K + B}.$$

More precisely, fix a top rational differential form $\omega \in \wedge^{\dim(X)} \Omega^1_{k(X)/k}$ with $K = (\omega)_X$. For a birational contraction $\mu: Y \to X$, the Weil divisor $(\omega)_Y$ is a canonical divisor of Y. Then $\mathbf{A}(X, B)_Y$ is the unique $\mathbb{Q}$-Weil divisor on Y such that the following adjunction formula holds:

$$\mu^*((\omega)_X + B) = (\omega)_Y - \mathbf{A}(X, B)_Y.$$

It is easy to see that $\mathbf{A}(X, B)$ is independent of the choice of ω and in fact it is independent of the choice of the canonical divisor K in its linear equivalence class.

A log pair (X, B) is said to have at most *Kawamata log terminal singularities* if $\text{mult}_E(\mathbf{A}(X, B)) > -1$ for every geometric valuation E. A log variety X has *canonical singularities* if $\text{mult}_E(\mathbf{A}(X)) \geq 0$ for every geometric valuation E which is exceptional on X.

2. Nef reduction

The existence of the nef reduction map is originally due to Tsuji [24]. An algebraic proof of the sharper statement below is due to Bauer, Campana, Eckl, Kebekus, Peternell, Rams, Szemberg and Wotzlaw [2].

Theorem 2.1. *[24,2] Let D be a nef $\mathbb{R}$-Cartier divisor on a normal projective variety X. Then there exists a rational map $f\colon X \dashrightarrow Y$ to a normal projective variety Y, satisfying the following properties:*

(i) f is a dominant rational map with connected fibers, which is a morphism over the general point of Y.

(ii) There exists a countable intersection U of Zariski open dense subsets of X such that for every curve C with $C \cap U \neq \emptyset$, $f(C)$ is a point if and only if $D \cdot C = 0$.

Moreover, $D|_W \equiv 0$ for general fibers W of f.

The rational map f is unique, and is called the *nef reduction of* D. The dimension of Y is called the *nef dimension* of D, denoted by $n(X, D)$. In general, the following inequalities hold [9,2]:

$$\kappa(X, D) \leq \nu(X, D) \leq n(X, D) \leq \dim(X).$$

Definition 2.2. *A nef $\mathbb{Q}$-Cartier divisor D is called good if*

$$\kappa(X, D) = \nu(X, D) = n(X, D).$$

Remark 2.3. This is equivalent to Kawamata's definition [9]. If

$$\kappa(X, D) = \nu(X, D),$$

there exists a dominant rational map $f\colon X \dashrightarrow Y$ and a nef and big $\mathbb{Q}$-divisor H on Y such that $\overline{D} \sim_{\mathbb{Q}} f^*(\overline{H})$, by [9]. In particular, $n(X, D)$ coincides with the Iitaka and the numerical dimension in this case.

Remark 2.4. [2] The extremal values of the nef dimension are:

(i) $n(X, D) = 0$ if and only if D is numerically trivial ($\nu(X, D) = 0$).

(ii) $n(X, D) = \dim(X)$ if and only if there exists a countable intersection U of Zariski open dense subsets of X such that $D \cdot C > 0$ for every curve C with $C \cap U \neq \emptyset$.

3. Fujita decomposition

Definition 3.1. [6] *An $\mathbb{R}$-Cartier divisor D on a normal proper variety X has a* Fujita decomposition *if there exists a b-nef $\mathbb{R}$-b-divisor $\mathbf{P}$ of X with the following properties:*

(i) $\mathbf{P} \le \overline{D}$.
(ii) $\mathbf{P} = \sup\{\mathbf{H};\, \mathbf{H}$ *b-nef* $\mathbb{R}$*-b-divisor*, $\mathbf{H} \le \overline{D}\}$.

The $\mathbb{R}$-b-divisor $\mathbf{P} = \mathbf{P}(D)$ is unique if it exists, and is called the semi-positive part *of D. The $\mathbb{R}$-b-divisor $\mathbf{E} = \overline{D} - \mathbf{P}$ is called the* negative part *of D, and $\overline{D} = \mathbf{P} + \mathbf{E}$ is called the* Fujita decomposition *of D.*

Remark 3.2. Allowing divisors with real coefficients is necessary: there exist Cartier divisors (in dimension at least 3) which have a Fujita decomposition with irrational semi-positive part [3].

Clearly, a nef $\mathbb{R}$-Cartier divisor D has a Fujita decomposition, with semi-positive part $\overline{D}$. More examples can be constructed using the following property:

Proposition 3.3. [6] *Let $f\colon X \to Y$ be a proper contraction, let D be an $\mathbb{R}$-Cartier divisor on Y and let E be an effective $\mathbb{R}$-Cartier divisor on X such that E is vertical and supports no fibers over codimension one points of Y.*

Then D has a Fujita decomposition if and only if $f^*D + E$ has a Fujita decomposition, and moreover, $\mathbf{P}(f^*D + E) = f^*(\mathbf{P}(D))$.

Lemma 3.4. *Assume LMMP and Log Abundance. Let (X, B) be a log variety with log canonical singularities. Then $K + B$ has a Fujita decomposition if and only if $\kappa(X, K + B) \ge 0$, and the semi-positive part is semi-ample. Moreover,*

$$\mathbf{P}(K + B) = \overline{K_Y + B_Y},$$

for a log minimal model (Y, B_Y).

Proof. If $K + B$ is nef, it has a Fujita decomposition with semi-positive part $\overline{K + B}$. By Log Abundance, it is semi-ample. If $K + B$ is not nef, we run the LMMP for (X, B). We may assume that X is $\mathbb{Q}$-factorial by Proposition 3.3. If $f\colon (X, B) \to Y$ is a divisorial contraction, then

$$K + B = f^*(K_Y + B_Y) + \alpha E,$$

where E is exceptional on Y and $\alpha > 0$. Thus $K + B$ has a Fujita decomposition if and only if $K_Y + B_Y$ has, and the semi-positive parts coincide. If $t\colon (X, B) \dashrightarrow (X^+, B_{X^+})$ is a log-flip, then

$$\overline{K + B} = \overline{K_{X^+} + B_{X^+}} + \mathbf{E},$$

where $\mathbf{E}$ is an effective $\mathbb{Q}$-b-divisor which is exceptional on both X and X'. Therefore $K+B$ has a Fujita decomposition if and only if $K_{X^+}+B_{X^+}$ has, and the semi-positive parts coincide.

If $f\colon (X,B)\to Y$ is a log Fano fiber space, $K+B$ admits no Fujita decomposition. □

Lemma 3.5. [9,6] *Let $f\colon X\to Y$ be a contraction of normal proper varieties, and let D be a nef $\mathbb{R}$-divisor on X which is vertical on Y. Then there exists a b-nef $\mathbb{R}$-b-divisor $\mathbf{D}$ of Y such that $\overline{D}=f^*\mathbf{D}$.*

Proof. After a resolution of singularities, Hironaka's flattenining and the normalization of the total space of the induce fibration, we have a fiber space induced by birational base change

$$\begin{array}{ccc} X & \xleftarrow{\mu} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \longleftarrow & Y' \end{array}$$

such that f' is equi-dimensional, X' is normal and Y' is non-singular, and μ^*D is vertical on Y'. Let D' be the largest $\mathbb{R}$-divisor on Y' such that $f'^*D'\le\mu^*D$. Since f' is equi-dimensional, $E=\mu^*D-f'^*D'$ is effective and supports no fibers over codimension one points of Y'. Furthermore, E is f'-nef since D is nef. By [6, Lemma 1.5], $E=0$. Therefore $\mu^*D=f'^*(D')$. In particular, D' is nef and $\mathbf{D}=\overline{D'}$ satisfies the required properties. □

4. Parabolic fiber spaces

We recall results of Kawamata [8,10] and Fujino, Mori [4,5] on adjunction formulas of Kodaira type for parabolic fiber spaces. These results are best expressed through Shokurov's terminology of b-divisors. With a view towards the logarithmic case, we introduce them via lc-trivial fibrations.

Definition 4.1. [1] *An* lc-trivial fibration $f\colon (X,B)\to Y$ *consists of contraction of normal varieties $f\colon X\to Y$ and a log pair (X,B), satisfying the following properties:*

(1) (X,B) *has Kawamata log terminal singularities over the generic point of Y.*
(2) rank $f_*\mathcal{O}_X(\lceil\mathbf{A}(X,B)\rceil)=1$.
(3) There exist a positive integer r, a rational function $\varphi\in k(X)^\times$ and a $\mathbb{Q}$-Cartier divisor D on Y such that

$$K+B+\frac{1}{r}(\varphi)=f^*D.$$

A *parabolic fiber space* is a contraction of non-singular proper varieties $f\colon X \to Y$ such that the generic fiber F has Kodaira dimension zero. Let b be the smallest positive integer with $|bK_F| \neq \emptyset$. We fix a rational function $\varphi \in k(X)^\times$ such that $K + \frac{1}{b}(\varphi)$ is effective over the generic point of Y.

Lemma 4.2. *Let $f\colon X \to Y$ be a parabolic fiber space. Then there exists a unique $\mathbb{Q}$-divisor B_X on X satisfying the following properties:*

(i) $K_X + B_X + \frac{1}{b}(\varphi) = f^*D$ for some $\mathbb{Q}$-divisor D on Y.
(ii) There exists a big open subset $Y^\dagger \subseteq Y$ such that $-B_X|_{f^{-1}(Y^\dagger)}$ is effective and contains no fibers of f in its support.

In particular, $f\colon (X, B_X) \to Y$ is an lc-trivial fibration.

Proof. See [4] for the existence and uniqueness of B_X. It remains to verify that $f\colon (X, B_X) \to Y$ is an lc-trivial fibration. The adjunction formula (3) is exactly (i). Let (F, B_F) be the induced log pair structure on a general fibre F of f. Since F is smooth and $B_F \le 0$, it is clear that (F, B_F) has Kawamata log terminal singularities, i.e. (1) holds.

As for (2), note first that given a resolution of singularities $\mu\colon X' \to X$, the induced fibration $X' \to Y$ is also parabolic and $\mu^*(K_X + B_X) = K_{X'} + B_{X'}$. Therefore we may assume that B_F has simple normal crossings support, in which case (2) becomes

$$\dim H^0(F, \lceil -B_F \rceil) = 1.$$

Since $0 \le -B_F \le \lceil -B_F \rceil$, we have $\dim H^0(F, \lceil -B_F \rceil) \ge 1$. Since F has zero Kodaira dimension and $-B_F \sim_{\mathbb{Q}} K_F$, we infer $\kappa(F, -B_F) = 0$. The effectivity of $-B_F$ implies that there exists a positive rational number t such that $-B_F \le \lceil -B_F \rceil \le t(-B_F)$. Therefore $\kappa(F, \lceil -B_F \rceil) = 0$, hence $\dim H^0(F, \lceil -B_F \rceil) \le 1$. We conclude that (2) holds. □

Definition 4.3. *Let $f\colon X \to Y$ be a parabolic fiber space with a choice of a rational function φ, as above. The* moduli $\mathbb{Q}$-b-divisor *of f, denoted $\mathbf{M} = \mathbf{M}(f, \varphi)$, is the moduli $\mathbb{Q}$-b-divisor of the lc-trivial fibration $f\colon (X, B_X) \to Y$.*

If φ' is another choice of the rational function, then $b\mathbf{M}(f, \varphi) \sim b\mathbf{M}(f, \varphi')$. Therefore $b\mathbf{M}$ is uniquely defined up to linear equivalence. According to the following Lemma, $\mathbf{M}$ is independent of birational changes of f:

Lemma 4.4. *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "\nu"'] \arrow[d, "f'"] \\ Y & Y' \arrow[l, "\mu"'] \end{tikzcd}$$

where f, f' are parabolic fiber spaces and μ, ν are birational contractions. Then $\mathbf{M}(f) = \mathbf{M}(f')$.

Proof. Assume first that μ is the identity morphism. Since X, X' are nonsingular, it is easy to see that $\mathbf{A}(X, B_X) = \mathbf{A}(X', B_{X'})$. Therefore $\mathbf{M}(f) = \mathbf{M}(f')$.

We are left with the case when ν is the identity morphism. Let $B_X^{(Y)}$ and $B_X^{(Y')}$ be the $\mathbb{Q}$-divisors induced by f and f', respectively. Since the general fiber is non-singular of zero Kodaira dimension, there exists a $\mathbb{Q}$-divisor C on Y' such that $B_X^{(Y')} = B_X^{(Y)} + f'^*C$. Therefore $\mathbf{M}(f) = \mathbf{M}(f')$, by [1, Remark 3.3]. $\square$

Theorem 4.5. *Let $f\colon X \to Y$ be a parabolic fiber space.*

(1) Consider a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "\nu"'] \arrow[d, "f'"] \\ Y & Y' \arrow[l, "\varrho"'] \end{tikzcd}$$

where ϱ is a proper surjective morphism, and f' is an induced parabolic fiber space. Then $\varrho^*\mathbf{M}(f) \sim_{\mathbb{Q}} \mathbf{M}(f')$.

(2) If f is semi-stable in codimension one, then

$$f_*\mathcal{O}_X(iK_{X/Y})^{**} = \mathcal{O}_Y(i\mathbf{M}_Y) \cdot \varphi^{\frac{i}{b}}, \text{ for } b|i.$$

(3) The moduli $\mathbb{Q}$-b-divisor $\mathbf{M}(f)$ is b-nef.

The key result of this section is the following corollary of [10, Theorem 3.6]:

Theorem 4.6. *Let $f\colon X \to Y$ be a parabolic fiber space. Assume that its geometric generic fiber $X \times_Y \mathrm{Spec}(\overline{k(Y)})$ is birational to a normal variety $\bar{F}$ with canonical singularities, defined over $\overline{k(Y)}$, such that $K_{\bar{F}}$ is semi-ample. Then the moduli $\mathbb{Q}$-b-divisor $\mathbf{M}(f)$ is b-nef and good.*

Proof. From the definition of the variation of a fiberspace, there exists a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & \bar{X} \arrow[l] \arrow[r] \arrow[d, "\bar{f}"'] & X^! \arrow[d, "f^!"'] \\ Y & \bar{Y} \arrow[l, "\tau"'] \arrow[r, "\varrho"] & Y^! \end{tikzcd}$$

such that the folowing hold:

(1) $\bar{f}$ and $f^!$ are parabolic fiber spaces.
(2) τ is generically finite, and ϱ is a proper dominant morphism.
(3) $\bar{f}$ is birationally induced via base change by both f and $f^!$.
(4) $\mathrm{Var}(f) = \mathrm{Var}(f^!) = \dim(Y^!)$.

Let $\mathbf{M}$, $\bar{\mathbf{M}}$, $\mathbf{M}^!$ be the corresponding moduli $\mathbb{Q}$-b-divisors. After a generically finite base change, we may also assume that $\mathbf{M}^!$ descends to $Y^!$, and $f^!$ is semi-stable in codimension one. By (3) and Theorem 4.5, we have

$$\tau^*\mathbf{M} = \bar{\mathbf{M}} \sim_{\mathbb{Q}} \varrho^*(\mathbf{M}^!).$$

In particular, $\kappa(\mathbf{M}) = \kappa(\mathbf{M}^!)$. Since $\bar{F}$ is a good minimal model, Viehweg's $Q(f^!)$ Conjecture holds [10, Theorem 1.1.(i)], that is the sheaf $(f^!_*(\omega^{\otimes i}_{X^!/Y^!}))^{**}$ is big for i large and divisible. But $(f^!_*(\omega^{\otimes i}_{X^!/Y^!}))^{**} \simeq \mathcal{O}_{Y^!}(i\mathbf{M}^!_{Y^!})$ for $b|i$, since $f^!$ is semi-stable in codimension one. Equivalently, $\kappa(Y^!, \mathbf{M}^!_{Y^!}) = \dim(Y^!)$, or $\mathbf{M}^!$ is b-nef and big. Therefore $\tau^*\mathbf{M}$ is b-nef and good, hence $\mathbf{M}$ is b-nef and good. □

5. Reduction argument

Theorem 5.1. *Let X be a projective variety with canonical singularities such that the canonical divisor K is nef. If $n(X, K) \leq 3$, then the canonical divisor K is semi-ample.*

Proof. Let $\Phi\colon X \dashrightarrow Y$ be the quasi-fibration associated to the nef canonical divisor K of X, and let Γ be the normalization of the graph of Φ:

$$\begin{tikzcd} & \Gamma \arrow[dl, "\mu"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

Since Φ is a quasi-fibration, μ is birational, f is a contraction and $\mathrm{Exc}(\mu) \subset \Gamma$ is vertical over Y. Let W be a general fiber of f.

Step 1: W is a normal variety with canonical singularities, and $K_W \sim_{\mathbb{Q}} 0$. Indeed, W has canonical singularities and $K_W = \mu^*K|_W$. The definition of Φ implies that K_W is numerically trivial. From [10, Theorem 8.2], we conclude that $K_W \sim_{\mathbb{Q}} 0$.

Step 2: There exist a diagram

$$\begin{tikzcd} X & X' \arrow[l, "\mu"'] \arrow[d, "f'"] \\ & Y' \end{tikzcd}$$

satisfying the following properties:

(a) μ is a birational contraction.
(b) $f'\colon X' \to Y'$ is a parabolic fiber space.
(c) There exists a simple normal crossings divisor Σ on Y' such that f' is smooth over $Y' \setminus \Sigma$.

(d) The moduli $\mathbb{Q}$-b-divisor $\mathbf{M} = \mathbf{M}(f')$ descends to Y' and there exists a contraction $h\colon Y' \to Z$ and a nef and big $\mathbb{Q}$-divisor N on Z such that $\mathbf{M}_{Y'} \sim_{\mathbb{Q}} h^*N$.
(e) Let E be any prime divisor on X'. If E is exceptional over Y', then E is exceptional over X.

Indeed, we may assume that Y is non-singular. Let $\Gamma' \to \Gamma$ be a resolution of singularities, and let $f_0\colon \Gamma' \to Y$ be the induced contraction. The general fiber of f_0 is birational to the general fiber of f. The latter is a normal variety W with canonical singularities, and $K_W \sim_{\mathbb{Q}} 0$. Therefore f_0 is a parabolic fiber space. We define $f'\colon X' \to Y'$ to be a parabolic fiber space induced after a sufficiently large birational base change $Y' \to Y$. By Theorem 4.6, the moduli $\mathbb{Q}$-b-divisor $\mathbf{M}(f) = \mathbf{M}(f_0)$ satisfies (d) once Y' dominates a certain resolution of Y. Also, (e) holds once f' dominates a flattening of f, and (b) follows from Hironaka's embedded resolution of singularities.

Step 3: There exists an effective $\mathbb{Q}$-divisor Δ on Y' such that (Y', Δ) is a log variety with Kawamata log terminal singularities, $K_{Y'} + \Delta$ has a Fujita decomposition and $\overline{K} \sim_{\mathbb{Q}} f'^*(\mathbf{P}(K_{Y'} + \Delta))$.

Indeed, the parabolic fiber space f' induces an lc-trivial fibration $(X', B_{X'}) \to Y'$, with associated discriminant divisor $B_{Y'}$. We have

$$K_{X'} + B_{X'} + \frac{1}{b}(\varphi) = f'^*(K_{Y'} + B_{Y'} + \mathbf{M}_{Y'}).$$

It is clear that $B_{Y'}$ is effective, $\lfloor B_{Y'} \rfloor = 0$ and $\operatorname{Supp}(B_{Y'}) \subseteq \Sigma$. Therefore $(Y', B_{Y'})$ is a log variety with Kawamata log terminal singularities. By (d), there exists an effective $\mathbb{Q}$-divisor Δ on Y' such that (Y', Δ) is a log variety with Kawamata log terminal singularities, and $\Delta \sim_{\mathbb{Q}} B_{Y'} + \mathbf{M}_{Y'}$. In particular,

$$K_{X'} + B_{X'} \sim_{\mathbb{Q}} f'^*(K_{Y'} + \Delta).$$

Let $\mu^*K = K_{X'} - A$ and let $B_{X'} = E^+ - E^-$ be the decomposition into positive and negative parts. It is clear that A is effective and exceptional over X, and $A - E^-$ is vertical on Y. Thus there exist effective $\mathbb{Q}$-divisors $A' \leq A$ and $E' \leq E^-$ such that $A - E^- = A' - E'$ and E' is vertical and supports no fibers over codimension one points of Y'. In particular,

$$\mu^*K + A' + E^+ \sim_{\mathbb{Q}} f'^*(K_{Y'} + \Delta) + E'.$$

By (e), the left hand side has a Fujita decomposition, with semi-positive part $\overline{K}$. Proposition 3.3 applies, hence $K_{Y'} + \Delta$ has a Fujita decomposition and $\overline{K} \sim_{\mathbb{Q}} f'^*(\mathbf{P}(K_{Y'} + \Delta))$.

Step 4: From the LMMP and Abundance applied to the log variety (Y', Δ), the semi-positive part of $K_{Y'} + \Delta$ is b-semi-ample. Therefore $\overline{K}$ is b-semi-ample, that is K is a semi-ample $\mathbb{Q}$-divisor. □

6. Divisors of maximal nef dimension which are not big

We prove Theorem 0.3 in this section. We fix the notation: X is a smooth projective surface and D is a nef Cartier divisor which has maximal nef dimension, but it is not big. We denote by K the canonical divisor of X.

Proposition 6.1. *The following hold:*

(1) $\kappa(X, D) \leq 0$, $\nu(X, D) = 1$.
(2) $D \cdot K \geq 0$.
(3) If $D \cdot K = 0$, *one of the following holds:*
  a) $\kappa(X, D) = -\infty$ *and* X *is birational to* $\mathbb{P}_C(\mathcal{E})$, *where* C *is a non-rational curve.*
  b) $\kappa(X, D) = 0$ *and* X *is either a rational surface, or an elliptic ruled surface.*
(4) Assume $D \cdot K = 0$ *and* $K^2 \geq 0$. *Then* D *is algebraically equivalent to an effective divisor.*

Proof. Since D cannot be good, (1) holds. We have

$$\chi(X, mD) = \frac{-D \cdot K}{2} m + \chi(\mathcal{O}_X).$$

Since $\nu(X, D) > 0$, $h^2(mD) = h^0(K - mD) = 0$ for $m \gg 0$.

(2) If $D \cdot K < 0$, then $\kappa(X, D) \geq 1$. This contradicts (1).

(3) Assume $D \cdot K = 0$. In particular, $\kappa(X) \leq 0$. Indeed, let L be a divisor such that $DL = 0$. Since D is nef, D is orthogonal on the irreducible components of all divisors in $|mL|$, $m \geq 0$. Since D is orthogonal on at most a countable number of curves, $\kappa(X, L) \leq 0$.

Assume $\kappa(X) = 0$. Let $\sigma \colon X \to X'$ be the birational contraction to a minimal model. Since $K_{X'} \sim_{\mathbb{Q}} 0$, $K \sim_{\mathbb{Q}} E$ where E is effective and $\mathrm{Supp}(E) = \mathrm{Exc}(\sigma)$. Since $D \cdot K = 0$, D is orthogonal on each exceptional divisor, hence $D = \sigma^*(D_{X'})$. Thus we may assume X is a minimal model. After an étale cover, X is an Abelian surface or a $K3$ surface. If X is an Abelian surface, D is big by the same argument as in [20, Proposition 1.4]. Contradiction. If X is a $K3$ surface, $h^0(X, mD) = h^1(X, mD) + 2$ by Riemann-Roch, hence $\kappa(X, D) \geq 1$. Contradiction.

Therefore $\kappa(X) = -\infty$. Riemann-Roch gives

$$h^0(X, mD) = h^1(X, mD) + 1 - q(X), \quad m \geq 1$$

If $q(X) = 0$, then $h^0(D) > 0$. We are in case (b), and the rest of the claim is well known (see [19]). Assume $q(X) > 0$. Then there exists a birational contraction $X \to X' = \mathbb{P}_C(E)$, with $q(X) = g(C) \geq 1$. We are in case (a).

(4) If $q(X) = 0$, $|D| \neq \emptyset$ by Riemann-Roch. Assume $q(X) > 0$. There exists a birational contraction $X \to X' = \mathbb{P}_C(E)$, with $q(X) = g(C)$. Since

$0 \le K_X^2 \le K_{X'}^2 = 8(1-q(X)) \le 0$, we infer that $X = \mathbb{P}_C(\mathcal{E})$ and C is an elliptic curve, i.e. $q(X) = 1$.

If $h^1(D + F_t - F_0) > 0$ for some $t \in C$, then $h^0(D + F_t - F_0) > 0$ by Riemann-Roch. Assume $h^1(D+F_t-F_0) = 0$ for every $t \in C$. Since D is of maximal nef dimension, $D \cdot F_0 > 0$. Therefore $h^0(F_0, D|_{F_0}) > 0$. By [20], Proposition 1.5, D is algebraically equivalent to an effective divisor. □

Theorem 6.2. *[19] In the case (3b) above, assume moreover that D is effective and $DC > 0$ for every (-1)-curve C of X. Then the pair (X, D) is classified as follows:*

(i) X is a rational surface such that $-K$ is nef and $K^2 = 0$. There exists a connected effective cycle $\sum n_i C_i \in |-K|$ such that the greatest common divisor of the n_i's is 1. Also, $D = m \sum n_i C_i$ for some positive integer m.

(ii) $X = \mathbb{P}_C(\mathcal{E})$ is a geometrically ruled surface over an elliptic curve C, of type II_c or II_c^* in Sakai's classification table:

a) $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{O}_C(d)$ with $d \in \mathrm{Pic}^0(C)$ non-torsion. Let C' be the section with $C' \sim C_0 - \pi^* d$. Then $K + C_0 + C' = 0$ and $D = d_0 C_0 + d' C'$.

b) $\mathcal{E}$ is an indecomposable extension of $\mathcal{O}_C$ by $\mathcal{O}_C$, $K + 2C_0 = 0$ and $D = d_0 C_0$.

Proof. (of Theorem 0.3) Contracting all (-1)-curves on which D is numerically trivial, we obtain a birational contraction $f : X \to Y$ such that $D = f^*(D_Y)$ and $A = K - f^*(K_Y)$ is effective, exceptional on Y.

In particular,

$$\kappa(X, K + tD) = \kappa(Y, K_Y + tD_Y) \text{ for } t \in \mathbb{R}.$$

By construction, D_Y is positive on every K_Y-negative extremal ray of Y. Note that Y is not a Del Pezzo surface: otherwise D_Y is semi-ample, hence good, by the Base Point Free Theorem [11]. Therefore $-K_Y \cdot R \le 1$ for every K_Y-negative extremal ray R of Y. Moreover, $D_Y \cdot R \ge 1$ since D_Y is Cartier. Therefore $K_Y + tD_Y$ is nef for $t \ge 2$.

In particular,

$$(K_Y + tD_Y)^2 = K_Y^2 + 2(K_Y \cdot D_Y)t + (D_Y^2)t^2 \ge 0 \text{ for } t \ge 2.$$

Therefore either $(K_Y + tD_Y)^2 > 0$ for $t > 2$ (case (1)), or $K_Y^2 = K_Y \cdot D_Y = D_Y^2 = 0$. Assume the latter holds. By Proposition 6.1.(4), D_Y is algebraically equivalent to an effective divisor D'. The pairs (Y, D') are classified by Theorem 6.2. Exactly one of the following holds:

(i) Y is a rational surface and there exists $m \in \mathbb{N}$ such that $K_Y + \frac{1}{m} D_Y \equiv 0$.

(ii) $Y = \mathbb{P}_C(\mathcal{E})$, where C is an elliptic curve and $\deg(\mathcal{E}) = 0$, and $K_Y + tD_Y \equiv 0$ for some $0 < t \le 2$.

□

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