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ON MINIMAL LOG DISCREPANCIES

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Abstract. We propose a stronger form of the boundedness of minimal log discrepancies conjectured by V.V. Shokurov. This stronger form holds up to dimension 3 and for toric varieties, and is equivalent to the lower semi-continuity of minimal log discrepancies.

0. Introduction

A log variety (X, B) is a normal variety X equipped with an effective $\mathbb{R}$-divisor B such that $K + B$ is $\mathbb{R}$-Cartier, where K is the canonical divisor of X. For any Grothendieck point $\eta \in X$, one defines the *minimal log discrepancy of* (X, B) *at* η, denoted $a(\eta; B)$ [Sh88]. This is an invariant of the singularity of X at η, and is either $-\infty$, or a real number. For instance, $a(\eta; B) = \operatorname{codim} \eta$ if $B = 0$ and X is nonsingular at η. The main result of this paper is the following:

Theorem 0.1. *Let (X, B) be a log variety. Assume that either X is a toric variety and B is an invariant $\mathbb{R}$-divisor, or $\dim X \le 3$. If $\eta, \xi \in X$ are two Grothendieck points such that $\eta \in \bar{\xi}$, then*

$$a(\eta; B) \le a(\xi; B) + \operatorname{codim}(\eta, \xi).$$

This motivates us to propose the following:

Conjecture 0.2. *Let (X, B) be a log variety. If $\eta, \xi \in X$ are two Grothendieck points such that $\eta \in \bar{\xi}$, then $a(\eta; B) \le a(\xi; B) + \operatorname{codim}(\eta, \xi)$.*

We also introduce new invariants of log pairs, the *mld-spectrum* and the *mld-stratification*; we show that the former is a finite set and the latter is constructible (Theorem 2.3). Consequently, we obtain an equivalence between Conjecture 0.2 and Conjecture 2.4, on the lower semi-continuity of minimal log discrepancies. Roughly, the latter states that minimal log discrepancies can only decrease in special points (for instance, if $(X, 0)$ is the germ of a surface Du Val singularity, then the minimal log discrepancy at every closed point of X is 2, except at 0, where it drops to 1).

The particular case $\xi = \eta_X$ of Conjecture 0.2 is the boundedness of minimal log discrepancies conjectured by V.V.Shokurov [Sh88], that is $a(\eta; B) \le \operatorname{codim} \eta$ for every Grothendieck point η in X. It is known in fact that boundedness holds

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in the cases covered by Theorem 0.1 (cf., [Rd80, Mrk96, Ka93, Br97]), but we hope that our equivalent conjectures 0.2 and 2.4 will shed light on its general case.

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1. Preliminary

A *variety* is a reduced irreducible scheme of finite type over a fixed field k, of characteristic 0. We denote by η_X the generic point of a variety X. A Grothendieck point $\eta \in X$ is called *proper* if $\eta \neq \eta_X$. A neighborhood of η in X is an open subset $U \subseteq X$ such that $\eta \in U$. An *extraction* is a proper birational morphism of normal varieties.

A *log pair* (X, B) is a normal variety X equipped with an $\mathbb{R}$-Weil divisor B such that $K + B$ is $\mathbb{R}$-Cartier. (X, B) is called a *log variety* if moreover, B is effective. A log pair (X, B) has *log nonsingular support* if X is nonsingular and $\operatorname{Supp}(B)$ is a divisor with normal crossings [KMM, 0-2-9]. A *log resolution* of a log pair (X, B) is an extraction $\mu : \tilde{X} \to X$ such that $\tilde{X}$ is nonsingular and $\operatorname{Supp}(\mu^{-1}(B)) \cup \operatorname{Exc}(\mu)$ is a divisor with normal crossings.

If (X, B) is a log pair and $\mu : \tilde{X} \to X$ is an extraction, the *log codiscrepancy divisor* of (X, B) on $\tilde{X}$ is the unique divisor $\tilde{B}$ on $\tilde{X}$ such that $\mu^*(K+B) = K_{\tilde{X}} + \tilde{B}$ and $\tilde{B} = \mu^{-1}B$ on $\tilde{X} \setminus \operatorname{Exc}(\mu)$. The identity $\tilde{B} = \sum_{E \subset \tilde{X}} (1 - a(E; X, B))E$ associates to each prime divisor E of $\tilde{X}$ a real number $a(E; X, B)$, called the *log discrepancy* of E with respect to (X, B). The invariant $a(E; X, B)$ depends only on the valuation defined by E on the field of rational functions of X, with center $c_X(E) = \mu(E)$. For simplicity, we write $a(E; B)$ for $a(E; X, B)$.

Definition 1.1. [Sh88] The *minimal log discrepancy* of a log pair (X, B) at a proper Grothendieck point $\eta \in X$ is defined as

$$a(\eta; X, B) = \inf_{c_X(E) = \eta} a(E; X, B),$$

where the infimum is taken after all prime divisors on extractions of X having η as a center on X. We set by definition $a(\eta_X; X, B) = 0$.

The log pair (X, B) has only *log canonical (Kawamata log terminal) singularities* if $a(\eta; B) \geq 0$ $(a(\eta; B) > 0)$ for every proper point $\eta \in X$. (X, B) has only *canonical (terminal) singularities* if $a(\eta; B) \geq 1$ $(a(\eta; B) > 1)$ for every point $\eta \in X$ of codimension at least 2.

Minimal log discrepancies on a log pair (X, B) are computed as follows (cf., [Ko92, 17.1.1]). If $\operatorname{codim} \eta = 1$, then $a(\eta; B) = 1 - b_\eta$, where b_η is the coefficient of B in $\bar{\eta}$. Assume now $\operatorname{codim} \eta \geq 2$, and let $(\tilde{X}, \tilde{B})$ be a log resolution with a normal crossings divisor $\sum_i E_i$ supporting the divisors $\mu^{-1}(\bar{\eta})$ and $\tilde{B}$. If (X, B)

has only log canonical singularities in some neighborhood of η, then $a(\eta; B) = \min_{c_X(E_i)=\eta} a(E_i; B) \in \mathbb{R}_{\geq 0}$. Otherwise, $a(\eta; B) = -\infty$.

Example 1.2. Under the same assumptions, $a(\eta; B) = -\infty$ if $\eta \in E$ is a proper point of a prime divisor E with $a(\eta_E; B) < 0$.

Example 1.3. Assume that (X, B) is a log pair with log nonsingular support, having only log canonical singularities at $\eta \in X$. Then $a(\eta; B)$ is attained on the exceptional divisor of the blow-up of X in η, that is

$$a(\eta; B) = \operatorname{codim} \eta - \operatorname{mult}_\eta B.$$

Lemma 1.4. [Sh91] *Assume (X, B) is a log variety, and X is nonsingular at η. Then $a(\eta; B) \leq \operatorname{codim} \eta$. Moreover, $a(\eta; B) \geq \operatorname{codim} \eta - 1$ iff $\operatorname{mult}_\eta B \leq 1$ and $a(\eta; B) = \operatorname{codim} \eta - \operatorname{mult}_\eta B$.*

2. The mld stratification

Proposition 2.1. *Assume $W \subset X$ is a closed irreducible subvariety and (X, B) is a log pair with only log canonical singularities at η_W. Then there exists an open subset U of X such that $U \cap W \neq \emptyset$ and*

$$a(x; B) = a(\eta_W; B) + \dim W$$

for every closed point $x \in W \cap U$.

Proof. Shrinking X near W, we may assume there exists a log resolution $\mu : (\tilde{X}, \tilde{B}) \to (X, B)$ with a normal crossings divisor $\cup_{i \in I} E_i$ on $\tilde{X}$ supporting $\tilde{B} = \sum_i (1 - a_i) E_i$ such that the following are satisfied:

i) $\mu^{-1}(W) = \bigcup_{i \in I_W} E_i$ for some subset $I_W \subseteq I$;
ii) $W \subseteq \mu(C)$ for each strata C of $\cup_{i \in I} E_i$;
iii) $\operatorname{codim}(C \cap \mu^{-1}(x), C) = \dim W$ for every strata C of $\cup_{i \in I} E_i$ dominating W, and every closed point $x \in W$.
iv) (X, B) has only log canonical singularities and $\dim W > 0$.

Note that $a(\eta_W; B) = \min_{i \in I_W} a_i$. Fix $x \in W$. Let $\eta \in \mu^{-1}(x)$. By *iii)*, $a(\eta; \tilde{B}) = \sum_{\eta \in E_i} a_i + \dim W + \operatorname{codim}(\eta, C \cap \mu^{-1}(x))$, where C is the minimal strata of $\cup_{i \in I} E_i$ containing η. Since all the a_i's are non-negative, $a(\eta; \tilde{B}) \geq a(\eta_W; B) + \dim W$. Taking infimum after all η's as above, we obtain $a(x; B) \geq a(\eta_W; B) + \dim W$.

Finally, let $k \in I_W$ be an index such that $a(\eta_W; B) = a_k$. Let η be the generic point of an irreducible component of $E_k \cap \mu^{-1}(x)$ of maximal dimension: $\operatorname{codim} \eta = \dim W + 1$. By *iii)* again, E_k is the minimal strata of $\cup_{i \in I} E_i$ containing η. Therefore $a(\eta; \tilde{B}) = a_k + \operatorname{codim} \eta - 1 = a(\eta_W; B) + \dim W$, and $a(x; B) \leq a(\eta_W; B) + \dim W$. ∎

Definition 2.2. Let (X, B) be a log pair. The *mld-spectrum* of (X, B) is defined as the set $\operatorname{Mld}(X, B) := \{a(\eta; B); \eta \in X\} \subset \{-\infty\} \cup \mathbb{R}$. We denote by a^o the map $X \to \operatorname{Mld}(X; B)$ $(x \mapsto a(x; B))$, defined on the closed points of X. The partition of X into the fibers of the map a^o is called the *mld-stratification* of (X,B).

Theorem 2.3. *Given a log pair (X,B), the mld-spectrum $\mathrm{Mld}(X,B)$ is a finite set, and the mld-stratification is constructible, i.e., all the fibers of the map a^o are constructible sets.*

Proof. Suffices to prove that $a^o|_W$ takes a finite number of values and its fibers are constructible subsets, for every closed subset $W \subseteq X$. There is nothing to prove if $\dim W = 0$, so let $\dim W > 0$. Let W_0 be an irreducible components of W. From Example 1.2 and Proposition 2.1, there exists an open subset $U_0 \subset X$ such that $U_0 \cap W_0 \neq \emptyset$, $a^o|_{U_0 \cap W_0}$ is constant, and U_0 does not intersect the other irreducible components of W. Since $W = (W \setminus U_0) \sqcup (W_0 \cap U_0)$, we are done by Noetherian induction. □

Conjecture 2.4. [Am99] *For any log variety (X,B), the function a^o is lower semi-continuous. That is, every closed point $x \in X$ has a neighborhood $x \in U \subseteq X$ such that $a(x;B) = \inf_{x' \in U} a(x';B)$.*

Proposition 2.5. *The two conjectures 2.4 and 0.2 are equivalent.*

Proof. Assume Conjecture 0.2 is valid, and let $x \in X$ be a closed point. By Theorem 2.3, we may shrink X such that $x \in \bar{C}$ for every irreducible component C of the fibers of the map a^o. For $x' \in X$, there exists a C such that $x' \in C$. Since $x \in \bar{C}$, we infer that $a(x;B) \leq a(\eta_C;B) + \dim \eta_C$. But $a(\eta_C;B) + \dim \eta_C = a(x';B)$, so we are done.

Assume Conjecture 2.4 is valid. According to Proposition 2.1, we may assume that $\eta = \{x\}$ is a closed point and $x \in \bar{\xi}$. Let U_x be a neighborhood of x such that $a(x;B) \leq a(x';B)$ for all $x' \in U_x$. Then $U_x \cap \bar{\xi} \subset \bar{\xi}$ is an open dense subset. From Proposition 2.1, there exists some $x' \in U_x \cap \bar{\xi}$ such that $a(x';B) = a(\xi;B) + \dim \xi$. Therefore $a(x;B) \leq a(\xi;B) + \dim \xi$. □

Remark 2.6. (V.V. Shokurov) Conjecture 2.4 is equivalent to the following lower semi-continuity in Grothendieck points: if (X,B) is a log variety, every Grothendieck point $\eta \in X$ has a neighborhood U such that $a(\xi;B) \geq a(\eta;B)$ for every Grothendieck point $\xi \in U$ with $\dim \xi \leq \dim \eta$.

Remark 2.7. If (X,B_X) and (Y,B_Y) are two log pairs, we denote by $(X \times Y, B_{X \times Y})$ the *product log pair*, i.e., the usual product with canonical Weil divisor $K_{X \times Y} = K_X \times Y + X \times K_Y$ and pseudoboundary $B_{X \times Y} = B_X \times Y + X \times B_Y$. Then $a(\eta \times \xi; B_{X \times Y}) = a(\eta;B_X) + a(\xi;B_Y)$ for Grothendieck points η and ξ on X and Y respectively. In particular, $\mathrm{Mld}(X \times Y, B_{X \times Y}) = \mathrm{Mld}(X,B_X) + \mathrm{Mld}(Y,B_Y)$.

3. Conjecture 0.2 up to codimension 3

We denote by $\mathcal{H}_c$ the Conjecture 0.2 with the extra assumption $\operatorname{codim}\eta = c$. Since minimal log discrepancies are preserved by passing to generic hyperplane sections, once $\mathcal{H}_c$ is valid for $c < d$, $\mathcal{H}_d$ is equivalent to the following particular case: if x is a closed point on the log variety (X, B) of dimension d, and C is a curve passing through x, then $a(x; B) \le a(\eta_C; B) + 1$.
We fix x, and shrink X to neighborhoods of x without further notice. We may assume $a(x; B) > 1$, which implies that (X, B) has only log canonical singularities (note that (X, B) might not be Kawamata log terminal).

Theorem 3.1. *$\mathcal{H}_c$ is valid for $c = 1, 2, 3$.*

Proof. $\mathcal{H}_1$ can be easily checked on curves. For $\mathcal{H}_2$, let $x \in C \subset X$ be as above, with $\dim X = 2$. Since $a(x; B) > 1$, X is nonsingular at x and $a(x; B) = 2 - \operatorname{mult}_x B$. In particular, $a(x; B) - (a(\eta_C; B) + 1) = \operatorname{mult}_C B - \operatorname{mult}_x B \le 0$.

For $\mathcal{H}_3$, let $x \in C \subset X$ be as above, with $\dim X = 3$. Assume first $a(\eta_C; B) \le 1$. From the Log Minimal Model Program (cf., [Ka92]), there exists a crepant extraction $\mu : (\tilde{X}, \tilde{B}) \to (X, B)$ such that $\tilde{B}$ is effective and there exists a prime divisor E on $\tilde{X}$ with $\mu(E) = C$ and $a(\eta_E; \tilde{B}) = a(\eta_C; B)$. Let η be the generic point of a curve in the fiber of $\mu|_E : E \to C$ over x. By $\mathcal{H}_2$, $a(x; B) \le a(\eta; \tilde{B}) \le a(\eta_E; \tilde{B}) + 1$.

Let now $a(\eta_C; B) > 1$. We may assume that $a(x; B) > 2$. By Lemmas 3.2 and 1.4, X is nonsingular at both x and η_C and $a(x; B) - (a(\eta_C; B) + 1) = \operatorname{mult}_C B - \operatorname{mult}_x B \le 0$. $\square$

Lemma 3.2. *Let x be a closed point on a log variety (X, B) of dimension 3. Then X is nonsingular point at x if $a(x; B) > 2$.*

Proof. We first show that X has $\mathbb{Q}$-factorial singularities. Indeed, from the Log Minimal Model Program we can find a $\mathbb{Q}$-factorialization $\mu : (\tilde{X}, \tilde{B}) \to (X, B)$, where $(\tilde{X}, \tilde{B})$ is a log variety again. Since $a(x; B) = \min_{\eta \in \mu^{-1}(x)} a(\eta; \tilde{B})$, we infer from $\mathcal{H}_1$ and $\mathcal{H}_2$ that $\dim \mu^{-1}(x) = 0$. Zariski's Main Theorem (cf., [Ha77, Exercises II.3.22, III.11.2]) implies that μ is an isomorphism over a neighborhood of x, hence X is $\mathbb{Q}$-factorial.

From the proof of Theorem 3.1, $a(\eta_C; B) > 1$ for every curve passing through x. By $\mathcal{H}_2$, (X, B) and X have only terminal singularities. If x is a singular point of X, then it is an isolated terminal singularity, hence $a(x; B) \le a(x; 0) = 1 + \frac{1}{r} \le 2$, where r is the index of K_X at x [Rd80, Mrk96, Ka93]. Contradiction! $\square$

The following characterization of cDV singularities is folklore. We include a sketch of its proof for completeness:

Proposition 3.3. *Assume (X, B) is a log variety of dimension 3 and let $x \in X$ be a closed point. Then $a(x; B) = 2$ iff exactly one of the following holds:*

i) *X is nonsingular at x and $\operatorname{mult}_x B = 1$.*
ii) *$x \notin \operatorname{Supp}(B)$ and X has a cDV singularity at x (cf., [Rd80]).*

Sketch of proof. By Lemma 1.4 we may assume that x is a singular point of X. By lower semi-continuity, (X, B) has only canonical singularities. Assume first that B is $\mathbb{R}$-Cartier. By Lemma 3.2, $B = 0$ near x. According to [Rd80, 2.2], suffices to show that K_X is a Cartier divisor. If X has terminal singularities at x, then K_X is Cartier by [Rd80, Mrk96, Ka93]. Otherwise, let $\mu : \tilde{X} \to X$ be an extraction such that $\mu^* K_X = K_{\tilde{X}}$ and $\tilde{X}$ has terminal singularities ([Rd83, 0.6]). The terminal subcase implies that $K_{\tilde{X}}$ is Cartier near $\mu^{-1}(x)$, thus K_X is Cartier by the Contraction Theorem.

Assume now that B is not $\mathbb{R}$-Cartier. Let $\mu : (\tilde{X}, \tilde{B}) \to (X, B)$ be a small extraction such that $\tilde{X}$ is $\mathbb{Q}$-factorial. Then $\mu^{-1}(x)$ is a union of curves, none of them included in the support of $\tilde{B}$. In particular, $-K_{\tilde{X}}$ is μ-nef, but not μ-trivial. However, $\tilde{X}$ admits no flipping contraction: its *difficulty* [Sh86] is 0 since it has only terminal Gorenstein singularities. Contradiction! □

4. Toric minimal log discrepancies

We refer the reader to [Fu93] for definitions and basic notations of toric geometry. Let $X = T_N emb(\Delta)$ be a toroidal embedding, and let $\{B_i\}_{i=1}^r$ be the T_N-invariant divisors of X, corresponding to the primitive vectors $\{v_i\}_{i=1}^r$ on the 1-dimensional faces of Δ. Let $B = \sum_i (1 - a_i) B_i$ be an invariant $\mathbb{R}$-divisor such that $K + B$ is $\mathbb{R}$-Cartier. Equivalently, there exists a linear form $\varphi \in M_{\mathbb{R}}$ such that $\varphi(v_i) = a_i$ for every i. We may assume the log variety (X, B) has only log canonical singularities, i.e., $0 \le a_i \le 1$ for every i.

Under the above assumptions, we have the following formula for the minimal log discrepancies of (X, B) at the generic points of the orbits (cf., [Br97]):

$$a_\sigma := a(\eta_{\mathrm{orb}(\sigma)}; B) = \inf\{\varphi(v); v \in \mathrm{rel\ int}(\sigma) \cap N\},\ \sigma \in \Delta.$$

Here, rel int(σ) denotes the relative interior of $\sigma \subset \mathbb{R}\sigma$, and orb$(\sigma)$ is the T_N-orbit corresponding to the cone $\sigma \in \Delta$. Conjecture 2.4 for toric varieties follows from the following:

Theorem 4.1. *In the above notations, let $X = \bigsqcup_{\sigma \in \Delta} \mathrm{orb}(\sigma)$ be the partition of X into T_N-orbits.*

i) Each strata in the mld-stratification is a union of orbits. In other words, $a(x; B) = a_\sigma + \mathrm{codim}(\sigma)$ for every cone $\sigma \in \Delta$ and every closed point $x \in \mathrm{orb}(\sigma)$.

ii) $a_\sigma + \mathrm{codim}(\sigma) \le a_\tau + \mathrm{codim}(\tau)$ for all cones $\tau, \sigma \in \Delta$ such that τ is a face of σ (i.e. $\mathrm{orb}(\sigma)$ is in the closure of $\mathrm{orb}(\tau)$).

Proof. *i)* : The equality holds for the generic closed point $x \in \mathrm{orb}(\sigma)$ from Proposition 2.1. This extends to all the points in orb(σ) since T_N acts transitively on orbits and leaves the boundary fixed.

ii) : Let τ be a proper face of σ and let $a_\tau = \varphi(v)$ for some $v \in \mathrm{rel\ int}(\tau) \cap N$. There exist primitive vectors $v_{i_1}, \dots, v_{i_c}$ $(c = \mathrm{codim}(\tau, \sigma))$ on the 1-dimensional faces of σ such that $w = v + v_{i_1} + \cdots + v_{i_c} \in \mathrm{rel\ int}(\sigma)$. Therefore $a_\sigma \le \varphi(w) = \varphi(v) + a_{i_1} + \cdots + a_{i_c} \le a_\tau + \mathrm{codim}(\tau, \sigma)$. □

V.V. Shokurov also conjectured the following nonsingularity criterion [Sh88]: if (X, B) is a log variety and $a(\eta; B) > \operatorname{codim} \eta - 1$, then X is nonsingular at η. If X is a toric variety and B is an invariant $\mathbb{R}$-divisor, this holds due to the following:

Proposition 4.2. *Let $\sigma \subset N_{\mathbb{R}}$ be a strongly rational polyhedral cone generated by the primitive vectors $v_1, \ldots, v_r \in N$. Assume $\varphi \in M_{\mathbb{R}}$ is a linear form such that $0 \leq \varphi(v_i) \leq 1$ for every i, and let*

$$\varphi_\sigma := \inf\{\varphi(v); v \in \mathrm{rel\ int}(\sigma) \cap N\}.$$

If $\varphi_\sigma > \dim \sigma - 1$ then σ is a nonsingular cone.

Sketch of proof. We use induction on $n = \dim \sigma$. By lower semi-continuity, φ has the same property when restricted to any proper face of σ. In particular, every proper face of σ is nonsingular.

If σ is simplicial, i.e., $r = n$, $\varphi_\sigma \leq \frac{n}{2}$ unless σ is a nonsingular cone (cf., [Br97]). If σ is not simplicial, one may assume $r = n + 1$. This implies $v_{n+1} = v_1 + \ldots + v_s - v_{s+1} - \ldots - v_k$, where $s \geq 1$ and $s + 1 \leq k \leq n$. Therefore $\varphi_\sigma \leq \varphi(v_1 + \ldots + v_s + v_{k+1} + \ldots + v_n) \leq s + n - k \leq n - 1$. Contradiction! □

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