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Florin Ambro

Inversion of adjunction for non-degenerate hypersurfaces

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Abstract. We prove a precise inversion of adjunction formula for the log variety $(\mathbb{C}^{d+1}, X)$, where X is a non-degenerate hypersurface. As a corollary, the minimal log discrepancies of non-degenerate normal hypersurface singularities are bounded by dimension.

0. Introduction

A *log variety* (X, B) is a normal variety X endowed with an effective $\mathbb{R}$-Weil divisor B such that $K_X + B$ is $\mathbb{R}$-Cartier. For each point $P \in X$, the *minimal log discrepancy* $a(P; X, B)$ is an invariant of the singularity of the log variety (X, B) at P. In connection to the termination of flips in the Minimal Model Program, V.V. Shokurov conjectured certain spectral properties of minimal log discrepancies, in particular that they are bounded by the dimension of the variety [11]. This is known to hold if $\dim X \leq 3$, or if X is a toric variety and B is an invariant divisor [10, 9, 6, 3, 1]. Our main result adds to this list the case of non-degenerate hypersurface singularities:

Main Theorem. Let $0 \in X \subset \mathbb{C}^{d+1}$ be the germ of a normal, non-degenerate hypersurface singularity. Then

(i) $a(0; X) = a(0; \mathbb{C}^{d+1}, X)$.
(ii) $a(0; X) \leq d$, and equality holds if and only if X is nonsingular.

The first statement is a *precise* inversion of adjunction for the log variety $(\mathbb{C}^{d+1}, X)$. Inversion of adjunction [12] has been used by V.V. Shokurov in his construction of 3-fold flips, and is conjectured to hold for any log variety (see [7, 8]). The effective upper bound is an immediate corollary.

The proof of the above theorem is based on the special properties of non-degenerate hypersurface singularities. To each hypersurface singularity, one can associate a Newton fan Σ and a piecewise linear function φ on the lattice points of $|\Sigma|$, measuring the singularities of the pair $(\mathbb{C}^{d+1}, X)$ in toric valuations. Under

F. Ambro: Department of Mathematical Sciences, University of Tokyo, Komaba, Meguro-Ku, Tokyo 153-8914, Japan. e-mail: ambro@ms.u-tokyo.ac.jp

Current address: DPMMS, CMS, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, England. e-mail: f.ambro@dpmms.cam.ac.uk

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the non-degenerate assumption, the minimal log discrepancies of X and $(\mathbb{C}^{d+1}, X)$ are determined by φ, and Inversion of Adjunction is the property that φ attains its minimum inside some proper cone of the Newton fan (Proposition 2.1).

1. Preliminary

1-A. Log pairs, log discrepancies

Let (X, B) be a normal variety X endowed with an effective $\mathbb{R}$-Weil divisor B such that $K + B$ is $\mathbb{R}$-Cartier, where K is a canonical divisor of X. For a resolution of singularities $\mu: \tilde{X} \to X$ with exceptional locus $\mathrm{Exc}(\mu)$, there exists a unique divisor $\tilde{B}$ on Y such that $\mu^*(K_X + B) = K_{\tilde{X}} + \tilde{B}$ and $\tilde{B} = B$ on $\tilde{X} \setminus \mathrm{Exc}(\mu)$. The identity $\tilde{B} = \sum_{E \subset \tilde{X}} (1 - a(E; X, B))E$ associates to each prime divisor E of $\tilde{X}$ a real number $a(E; X, B)$, called the *log discrepancy* of E with respect to (X, B). The invariant $a(E; X, B)$ depends only on the valuation defined by E on the field of rational functions of X, with center $c_X(E) = \mu(E)$. We drop B from notation, in case it is zero.

We say that μ is a *log resolution* of (X, B) if $\mu^{-1}(\mathrm{Supp}(B)) \cup \mathrm{Exc}(\mu)$ is a divisor with simple normal crossings. The *minimal log discrepancy* of a log pair (X, B) at a point $P \in X$ is

$$a(P; X, B) = \inf_{c_X(E)=P} a(E; X, B),$$

where the infimum is taken after all prime divisors on resolutions of X, having P as a center on X [11]. The log pair (X, B) has only *log canonical singularities* if $a(E; X, B) \geq 0$ for every valuation E of X.

Minimal log discrepancies are computed as follows: let $(\tilde{X}, \tilde{B})$ be a log resolution such that $\mu^{-1}(P)$ is a divisor. Let $\cup_i E_i$ be the divisor with normal crossings supporting $\mathrm{Exc}(\mu)$ and $\tilde{B}$. If $\min_{c_X(E_i)=P} a(E_i; X, B) < 0$ then $a(P; X, B) = -\infty$. Otherwise, $a(P; X, B) = \min_{c_X(E_i)=P} a(E_i; B)$ is a non-negative real number and (X, B) has only log canonical singularities in a neighborhood of P.

1-B. The Newton polyhedron

To any hypersurface singularity

$$X: (f = 0) \subset \mathbb{C}^{d+1}$$

one can associate a fan, which is a subdivision of the standard fan [2]. We recall below this construction, and we also fix the notation (see [4] for standard toric notation and terminology). Fixing coordinates, we identify $\mathbb{C}^{d+1}$ with the toric variety $T_N(\sigma)$, where σ is the standard simplicial cone generated by $\{e_0, \dots, e_d\}$. The standard fan consists of the cones σ_I generated by $\{e_i; i \in I\}$, for every subset I of $\{0, \dots, d\}$. The dual cone $\sigma^\vee$ of σ is generated by the dual basis $\{e_0^*, \dots, e_d^*\}$. In this basis, $\sigma_I^\vee = \{m \in \sigma^\vee; m_i = 0\ \forall i \notin I\}$. The relative interior $\mathrm{relint}(\sigma)$ of a cone σ is its topological interior in the $\mathbb{R}$-vector space spanned by σ. A covector $a \in \sigma \setminus 0$ is *primitive* if $a \in N \cap \sigma$ and there exists no covector $a' \in N$ such that $a = ka'$ for some integer $k \geq 2$.

The *Newton polyhedron of* f, denoted Γ_+, is the convex hull of

$$\cup_{m \in \mathrm{Supp}(f)} (m + \sigma^\vee) \subset M_{\mathbb{R}}.$$

The *Newton diagram* Γ is the union of compact faces of Γ_+. The *supporting function* $l_\Gamma : \sigma \to [0, \infty)$ is defined as follows

$$l_\Gamma(a) = \min_{m \in \Gamma_+} (a, m).$$

The *trace* of a covector $a \in \sigma$ is $\mathrm{tr}(a) = \{m \in \Gamma_+; (a, m) = l_\Gamma(a)\}$. The faces of Γ_+ are the traces of covectors $a \in \sigma$. Compact faces are the traces of covectors $a \in \mathrm{relint}(\sigma)$, while non-compact faces are $\gamma + \sigma_I^\vee$, where γ is a compact face.

Two covectors $a, a' \in \sigma$ are said to be equivalent if they have the same trace. The closures of equivalence classes are closed cones forming the *Newton fan* $\Sigma_f = \Sigma_{\Gamma(f)}$, which is a subdivision of the standard fan. Each cone of the Newton fan can be represented as

$$\sigma_{\gamma,I} := \{a \in \sigma_I; (a, m) = l_\Gamma(a)\ \forall m \in \gamma\},\ \ \sigma_{\gamma,I} \cap \mathrm{relint}(\sigma_I) \neq \emptyset$$

where γ is a compact face of Γ and I is a subset of $\{0, \ldots, d\}$. We drop I from notation if $I = \emptyset$. The cones containing $\sigma_{\gamma,I}$ are $\{\sigma_{\tau,J}; \tau \prec \gamma,\ J \subset I\}$. We say that a cone of Σ_f is *proper* if it is not maximal dimensional. Note that the supporting function l_Γ is linear on each cone of the Newton fan: $l_\Gamma(a) = (a, m)$, for $a \in \sigma_\gamma$ and $m \in \gamma$.

Definition 1.1. *The power series* $f = \sum_m c_m x^m$ *is* non-degenerate *(with respect to its Newton polyhedron) if the hypersurfaces*

$$\left\{ \sum_{m \in \gamma} c_m x^m = 0 \right\} \subset (\mathbb{C} \setminus 0)^{d+1}$$

are non-singular for every compact face γ *of* Γ.

The key feature of hypersurface singularities defined by non-degenerate series is the existence of toric log resolutions:

Lemma 1.2. [2, 8.9] *Let* $X \subset \mathbb{C}^{d+1}$ *be a hypersurface given by a non-degenerate series* f. *Let* Σ *be a simple subdivision of the Newton fan* $\Sigma_{\Gamma(f)}$, *which contains the primitive covector* $\mathbf{1} = (1, \cdots, 1)$ *in its skeleton. Then the induced birational morphism* $\mu \colon T_N(\Sigma) \to \mathbb{C}^{d+1}$ *is a log resolution of* $(\mathbb{C}^{d+1}, X)$ *over a neighborhood of* 0, *and* $\mu^{-1}(0)$ *is a divisor.*

Due to this lemma, certain properties of the singularities of X can be read off the Newton polyhedron. For instance, X has log canonical singularities if and only if the vector $\mathbf{1} = (1, \ldots, 1)$ belongs to the Newton polyhedron Γ_+. This certainly fails to be true for arbitrary hypersurface singularities (see [5]).

Example 1.3. Let $X \subset \mathbb{C}^3$ be the surface singularity with equation

$$f = x^p + y^q + z^r - \alpha xyz,$$

where $p, q, r \geq 2$ and $\alpha \in \mathbb{C}$. Then f is non-degenerate with respect to its Newton polyhedron, with the following exceptions:

(1) $f = x^2 + y^3 + z^6 - \alpha xyz$, and $\alpha^6 = 432$.
(2) $f = x^2 + y^4 + z^4 - \alpha xyz$, and $\alpha^4 = 64$.
(3) $f = x^3 + y^3 + z^3 - \alpha xyz$, and $\alpha^3 = 27$.

Nonetheless, the singularities defined by (1), (2), (3) are log canonical (the strict transform of X on any toric resolution as in Lemma 1.2 has normal crossings, but not simple normal crossings, singularities). In fact, X has log canonical singularities if and only $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq 1$.

2. The log discrepancy function

Let $\mathrm{Shed}(\sigma^\vee)$ be the unit polyhedron $\{\sum_{i=0}^d t_i e_i^*; 0 \leq t_i \leq 1\}$. For $\delta \in \mathrm{Shed}(\sigma^\vee)$, the *log discrepancy function* is defined as

$$\varphi = \varphi_\delta : N \cap \sigma \to \mathbb{R}, \ \varphi(a) = (a, \delta) - l_\Gamma(a)$$

The function φ measures the singularities of the log variety

$$\left(\mathbb{C}^{d+1}, X + \sum_{i=0}^{d}(1 - \delta_i)H_i\right)$$

where $H_i \colon (x_i = 0)$ are the coordinate hyperplanes. Indeed, for a primitive covector $a \in \sigma \cap N$, let E_a be the exceptional divisor of the a-weighted blow up. Then $\varphi(a)$ is the log discrepancy of E_a with respect to the the log variety $(\mathbb{C}^{d+1}, X + \sum_{i=0}^d (1 - \delta_i)H_i)$. The original log variety $(\mathbb{C}^{d+1}, X)$ corresponds to the case when all components of δ are 1, but this slightly more general context is useful for the proof of our main theorem.

We assume until the end of this section that φ is *non-negative*, which is equivalent to $\delta \in \Gamma_+ \cap \mathrm{Shed}(\sigma^\vee)$. In particular, φ has a minimum.

Proposition 2.1. *The restriction* $\varphi \colon N \cap \mathrm{relint}(\sigma) \to \mathbb{R}$ *attains its minimum inside some proper cone of* Σ_f.

We first remark some basic properties of φ:

- $\varphi(ca) = c \cdot \varphi(a)$ for $c > 0$.
- $\varphi(a + a') \leq \varphi(a) + \varphi(a')$. Moreover, equality holds if and only if $\mathrm{tr}(a) \cap \mathrm{tr}(a') \neq \emptyset$.
- The zero locus $\mathcal{Z}(\varphi) := \{a \in \sigma; \varphi(a) = 0\}$ is a cone $\sigma_{\gamma,I}$ of Σ_Γ. Indeed, one can write (not uniquely) $\delta = \sum_i \lambda_i m^i + r$, where $\{m^i\}$ are the vertices of a compact face γ of Γ, $\lambda_i > 0\, \forall i$, $\sum_i \lambda_i = 1$, and $r \in \sigma^\vee$. Then $Z(\varphi) = \sigma_\gamma \cap r^\perp$. Note that $\delta \in \mathrm{relint}(\gamma + \sigma_I^\vee)$.

– The function φ can attain its minimum value only on cones containing $\mathcal{Z}(\varphi)$. Indeed, there exists $e \in \sigma_{\gamma,I} = \mathcal{Z}(\varphi)$ such that $\mathrm{tr}_\Gamma(e) = \gamma$. If $a \in \sigma \setminus \cup_{m\in\gamma}\sigma_m$, i.e. $\mathrm{tr}_\Gamma(a) \cap \gamma = \emptyset$, then $\varphi(a+e) < \varphi(a) + \varphi(e) = \varphi(a)$. Therefore φ cannot attain the minimum value at a.

The following lemma gives an algorithm for finding minimizing lattice points for the log discrepancy function:

Lemma 2.2. *Let $a \in \sigma \cap N$ such that $\mathrm{tr}_\Gamma(a) \cap \mathrm{tr}_\Gamma(e_j) = \emptyset$. Then $\varphi(a+e_j) \le \varphi(a)$, and equality holds if and only if $\delta_j = 1$ and one of the following holds:*

a) There exists a vertex m of Γ such that $a, a+e_j \in \sigma_m$, and $m_j = 1$. Note that $\min \varphi|_{\sigma_m}$ is attained on $\cup_{m'_j=0}\sigma_{mm'}$ in this case.
b) $m'_j = 0$ and $(a, m') = (a, m) + 1$ for every $m \in \mathrm{tr}_\Gamma(a)$ and $m' \in \mathrm{tr}_\Gamma(a+e_j)$. In particular, $e_j \in \sigma_{m'}$.

Proof. Let $m \in \mathrm{tr}_\Gamma(a)$, $m' \in \mathrm{tr}_\Gamma(a+e_j)$. Then

$$\varphi(a+e_j) - \varphi(a) = -(a, m'-m) + \delta_j - m'_j.$$

a) If there exists $m \in \mathrm{tr}_\Gamma(a) \cap \mathrm{tr}_\Gamma(a+e_j)$, then $\varphi(a+e_j) - \varphi(a) = \delta_j - m_j$. Since $e_j \notin \sigma_m$, $m_j \ge 1$, and there exists m' with $m'_j < m_j$. Thus $\varphi(a+e_j) \le \varphi(a)$, and equality holds if and only if $\delta_j = m_j = 1$.
b) Assume $\mathrm{tr}_\Gamma(a) \cap \mathrm{tr}_\Gamma(a+e_j) = \emptyset$. Then $\varphi(a+e_j) - \varphi(a) \le -1 + \delta_j - m'_j \le 0$, and equality holds if and only if $\delta_j = 1$, $m'_j = 0$, and $(a, m'-m) = 1$ for all $m \in \mathrm{tr}_\Gamma(a)$, $m' \in \mathrm{tr}_\Gamma(a+e_j)$. □

Proof of Proposition 2.1. Assume first that $I = \emptyset$. That is $\delta \in \Gamma$, and $\mathcal{Z}(\varphi) = \sigma_\gamma$ intersects $\mathrm{relint}(\sigma)$. If $\dim \gamma > 0$ then the minimum is attained only in the proper cone σ_γ. If $\dim \gamma = 0$, φ is identically zero on the maximal cone σ_γ.

Assume $I \neq \emptyset$. We may assume $\delta_i < 1$ for all $i \in I$. This is sufficient, since the desired property of $\varphi = \varphi_\delta$ is closed with respect to δ belonging to the convex polyhedron $\Gamma_+ \cap \mathrm{Shed}(\sigma^\vee)$. Assume by contradiction that minimum is not attained on proper cones. Let m be a vertex of γ such that φ attains its minimum inside the maximal cone σ_m. Since $\delta_i < 1$ for all $i \in I$, we obtain $e_i \in \sigma_m$ for all $i \in I$ by Lemma 2.2. In fact, we obtain $m_i = 0$ for all $i \in I$ since φ is non-negative and $\delta_i < 1$. By Lemma 2.2 again, φ attains minimum inside $\sigma_{m'}$ for some $m' \neq m$.

Let $m \in \gamma$ with $m_i = 0\ \forall i \in I$. Then the minimum of φ on $\sigma_m \cap \mathrm{relint}(\sigma)$ is attained on $\sigma_{\gamma,I} + \sum_{i\in I} e_i$. Indeed, for $a \in \sigma_m$ we have

$$\varphi(a) = \sum_{i\in I} \delta_i a_i + \varphi(\bar{a}) \ge \sum_{i\in I} \delta_i,$$

where $\bar{a}_i = a_i$ for $i \notin I$ and $\bar{a}_i = 0$ for $i \in I$. If $m, m' \in \gamma$ with $m_i = m'_i = 0\ \forall i \in I$ then $\sigma_{\gamma,I} + \sum_{i\in I} e_i \subset \sigma_{mm'}$ and we are done. □

3. Inversion of adjunction

Lemma 3.3. *Let $X\colon (f = 0) \subset \mathbb{C}^{d+1}$ be a hypersurface which does not contain any of the coordinate hyperplanes H_i. Let Σ be a simple subdivision of $\Sigma_{\Gamma(f)}$, and let $\mu\colon T_N(\Sigma) \to \mathbb{C}^{d+1}$ be the associated resolution. Let E_a be a μ-exceptional divisor centered at the origin, corresponding to a primitive covector $a \in \operatorname{relint}(\sigma) \cap N$. If E_a does not intersect the proper transform of X, then a belongs to a unique cone of $\Sigma_{\Gamma(f)}$.*

Proof. $T_N(\Sigma)$ is covered by the open sets $U_\tau \simeq \mathbb{C}^{d+1}$ corresponding to the maximal dimensional cones of Σ. Let $\tau \in \Sigma$ be a maximal cone such that $E_a \cap U_\tau \neq \emptyset$. The restriction $\mu : U_\tau \to \mathbb{C}^{d+1}$ can be identified with

$$\mu_\tau\colon \mathbb{C}^{d+1} \to \mathbb{C}^{d+1}, \quad x_i = y_0^{a_i^0} \cdots y_d^{a_i^d}$$

where $(a^0, \ldots, a^d)$ is the ordered skeleton of τ. We may assume $a = a^0$. Denote by X' the proper transform of X in $T_N(\Sigma)$. We have

$$\mu_\tau^*(f) = \sum_{m \,\in\, \operatorname{Supp}(f)} c_m \prod_{j=0}^{d} y_j^{(a^j,m)} = \left(\prod_{j=0}^{d} y_j^{l(a^j)}\right) f_\tau(y_0, \ldots, y_d),$$

where $f_\tau = 0$ is the equation of $X' \cap U_\tau$. The divisor E_a has equation $y_0 = 0$ in U_τ, thus $E_a \cap X' \cap U_\tau = \emptyset$ if and only if $f_\tau \equiv C \bmod y_0$ for some non-zero constant C. Equivalently, $\min_{m \in \operatorname{Supp}(f)}(a^0, m)$ is attained in exactly one point. Since $a \in \operatorname{relint}(\sigma)$, this implies that the trace of a is a vertex of Γ, i.e. a belongs to the interior of some maximal cone of Σ_Γ. $\square$

Proof of Main Theorem. (i): We may assume that X does not contain any of the coordinate hyperplanes. By Lemma 1.2, we obtain

$$a(0; X) \geq a(0; \mathbb{C}^{d+1}, X) = \inf \varphi|_{\operatorname{relint}(\sigma) \cap N}$$

where φ corresponds to $\delta = \mathbf{1}$. Assume that $a(0; \mathbb{C}^{d+1}, X) = -\infty$. Then φ takes negative values. In particular, there exists a primitive covector $a \in \operatorname{relint}(\sigma) \cap N$ contained in a proper cone of Σ_f such that $\varphi(a) < 0$. Let Σ be a simple subdivision of the fan Σ_Γ, such that a and $\mathbf{1}$ belong to its skeleton, and let $\mu\colon T_N(\Sigma) \to \mathbb{C}^{d+1}$ be the induced log resolution. By Lemma 3.3, the exceptional divisor E_a intersects the proper transform X' of X, and Lemma 1.2 implies that $a(0; X) \leq a(E_a \cap X'; X) = \varphi(a) < 0$. Therefore $a(0; X) = a(0; \mathbb{C}^{d+1}, X)$.

Assume that $a(0; \mathbb{C}^{d+1}, X) \geq 0$. In particular, φ is non-negative. By Proposition 2.1, there exists a primitive covector $a \in \operatorname{relint}(\sigma) \cap N$ contained in a proper cone of Σ_f such that $\varphi(a) = \min \varphi|_{\operatorname{relint}(\sigma) \cap N}$. The same argument as above implies that $a(0; X) \leq \varphi(a)$, hence $a(0; X) = a(0; \mathbb{C}^{d+1}, X)$.

(ii): If E is the exceptional divisor of the blow up of $\mathbb{C}^{d+1}$ at 0, then $a(E; \mathbb{C}^{d+1}, X) = d - (\operatorname{mult}_0(f) - 1)$. By (i), $a(0; X) = a(0; \mathbb{C}^{d+1}, X) \leq d$. It is clear that equality holds if and only if X is non-singular at 0. $\square$

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Note added in proof. Our main result has been recently generalized to normal hypersurface singularities, by Ein, Mustaţă and Yasuda, "Jet schemes, log discrepancies and Inversion of Adjunction" (math.AG/0209392).