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HABILITATION THESIS

(Counter)examples in Several Complex Variables

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ABSTRACT

Domeniul fundamental: Matematică şi ştiinţe ale naturii

Domeniul de abilitare: Matematică

Teză elaborată în vederea obţinerii atestatului de abilitare în scopul conducerii lucrărilor de doctorat în domeniul Matematică

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In this Habilitation Thesis I collect several examples that I obtained together with my coauthors since I obtained my Ph.D. They are all counterexamples to specific questions and they all belong to the field of Complex Analysis in Several Variables. I decided to include only the proofs that are strictly related to these examples. Hence the thesis does not contain the proofs of those results that might be of "independent interest". For the proofs of these later results I referred instead to the articles in which they appeared.

In Chapter 1 we collect a short list of definitions and results that are going to be used throughout the thesis. The main part is Chapter 2 which contains the original results of the thesis. According to the Romanian Ministry of Education guide for writing a Habilitation thesis, this thesis must contain plans and directions for future research. I decided to include, as a final chapter of the thesis, a short list of open problems. They are all of them difficult and most of them extremely difficult. I included them here not because I intend to solve them (not that I wouldn't want to) but because I believe that is always good to have in mind some difficult problems to which to relate our work.

We give below a brief description of Chapter 2. The results of the eight sections of this chapter are presented in the chronological order in which they were obtained. The motivation for constructing each example is given in the respective section.

Section 2.1 We give examples showing that the third Cauchy-Fantapiè integral formula of Leray does not hold in the form stated by Jean Leray in his 1959 paper "Problème de Cauchy III" and one needs to impose supplementary conditions. This example was obtained in a joint paper with Finnur Larusson that appeared in Mich. Math. J.in 2003.

Section 2.2. We give an example of a contractible Stein domain $D \subset \mathbb{C}^2$ with smooth real analytic boundary such that the intersection of D with every complex line $l \subset \mathbb{C}^2$ is Runge in l and D is not Runge in $\mathbb{C}^2$, thus answering a question of Bremermann. This example was given in a paper that appeared in Math. Ann. in 2007.

Section 2.3. For Riemann surfaces, Raghavan Narasimhan proved the following result: If $\{D_n\}_{n\geq 1}$ is a locally finite sequence of pairwise disjoint open subsets in an open Riemann surface, $\mathcal{R}$, such that $\cup D_n$ is Runge in $\mathcal{R}$ then for every sequence of holomorphic functions $f_n \in \mathcal{O}(D_n)$, every sequence of compacts $K_n \subset D_n$ and every sequence of positive numbers $\epsilon_n > 0$ there exists $f \in \mathcal{O}(\mathcal{R})$ such that $\|f - f_n\|_{K_n} < \epsilon_n$ for every n.

We give examples showing that a similar result does not hold in higher dimensions. These examples were given in a paper that appeared in J. Math. Kyoto Univ. in 2007.

Section 2.4. We construct a 1-convex complex manifold X of dimension 2 such that its universal covering $\tilde{X}$ has the following two properties:

a) $\tilde{X}$ does not satisfy the discrete disk property,
b) $H^1(\tilde{X}, \mathcal{O})$ is not separated.

The results of this section appeared in two joint articles with Mihnea Colţoiu that were published in Math Z. in 2013 and Adv. Math. in 2014.

Section 2.5. We give examples of complex spaces X such that the globally defined holomorphic functions on X separate points and give local coordinates and X is not biholomorphic to an open subset of a Stein space. These examples were obtained in a joint paper with Mihnea Colţoiu that appeared in Math. Ann. in 2013.

Section 2.6. We construct a sequence $\{D_\nu\}$ of 3-complete open subsets of $\mathbb{C}^5$ such that $D_{\nu+1} \subset D_\nu$ for every ν and the interior of their intersection $\mathrm{Int}\,(\bigcap D_\nu)$ is not cohomologically 3-complete.

For every integer $q \geq 2$, we construct a normal Stein complex space X with only one isolated singularity, and $\{D_\nu\}$ a decreasing sequence of open subsets of X such that each D_ν is 2-complete and $\mathrm{Int}\,(\bigcap D_\nu)$ is not q-complete with corners.

These examples were obtained in a joint paper with Mihnea Colţoiu that appeared in Publ. Res. Inst. Math. Sci. în 2017.

Section 2.7. We give an example of a polynomial map $F : \mathbb{C}^3 \to \mathbb{C}^2$ and a regular value $y_0 \in \mathbb{C}^2$ of F such that for y varying in a neighborhood of y_0 the fiber $F^{-1}(y)$ is homeomorphic to $F^{-1}(y_0)$ and F is not a C^∞ trivial fibration on a neighborhood of y_0. This example was obtained in a joint preprint with Mihai Tibăr.

Section 2.8. We give an example of a domain $\Omega \subset \mathbb{C}^n$ which is biholomorphic to a ball and such that for every Stein domain $U \subset \mathbb{C}^n$ containing $\overline{\Omega}$ we have that Ω is not Runge in U. This example was obtained in a joint preprint with Hervé Gaussier.