ACADEMIA ROMÂNĂ
SCOSAAR

HABILITATION THESIS

Submanifolds with Parallel Mean Curvature and Biharmonic Submanifolds in Riemannian Manifolds

Abstract

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Domeniul fundamental *Matematică și științe ale naturii*
Domeniul de abilitare *Matematică*

Teză elaborată în vederea obţinerii atestatului de abilitare în scopul conducerii lucrărilor de doctorat în domeniul *Matematică*

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Abstract

A classical but still very dynamic topic in the field of Differential Geometry is the study of *surfaces with constant mean curvature* (cmc surfaces) in 3-dimensional spaces and, more generally, of *submanifolds with parallel mean curvature vector field* (pmc submanifolds) in Riemannian manifolds with arbitrary dimension. Their history is spread on more than six decades, in the case of cmc surfaces, and goes back to the early 1970s, in the case of pmc submanifolds, to papers like [**35**] by B.-Y. Chen and G. D. Ludden, [**52, 53**] by J. Erbacher, [**58**] by D. Ferus, [**83**] by D. A. Hoffman, or [**136**] by S.-T. Yau.

In the following, we shall briefly recall only some of those results that represent a source of inspiration for our work. Two very powerful tools were mainly used to prove these results: *holomorphic differentials* defined on cmc (or pmc) surfaces and *Simons type equations*.

H. Hopf [**85**] was the first to use a holomorphic differential to show that any cmc surface homeomorphic to a sphere in Euclidean 3-space is actually a round sphere. His result was extended to cmc surfaces in 3-dimensional space forms by S.-S. Chern [**41**], and then to cmc surfaces in product spaces of type $M^2(c) \times \mathbb{R}$, where $M^2(c)$ is a complete simply-connected surface with constant curvature c, as well as $\mathrm{Nil}(3)$ and $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$, by U. Abresch and H. Rosenberg [**1, 2**].

The next natural step was to study pmc surfaces in product spaces of type $M^n(c) \times \mathbb{R}$, where $M^n(c)$ is a space form with constant sectional curvature c, i.e., those surfaces satisfying $\nabla^{\perp} H = 0$, where $\nabla^{\perp}$ is the connection in the normal bundle and H is the mean curvature vector field. Two very important papers on this topic are [**5, 6**] by H. Alencar, M. do Carmo, and R. Tribuzy. In these articles they introduce a holomorphic differential (that generalizes the Abresch-Rosenberg differential defined in [**1**] for cmc surfaces in $M^2(c) \times \mathbb{R}$) and use it to study the geometry of pmc surfaces. One of the main results in [**6**] is a reduction of codimension theorem, showing that a pmc surface immersed in $M^n(c) \times \mathbb{R}$ either is a minimal surface in a totally umbilical hypersurface of $M^n(c)$; or a cmc surface in a 3-dimensional totally umbilical or totally geodesic submanifold of $M^n(c)$; or it lies in $M^4(c) \times \mathbb{R}$.

Another very effective method developed in order to study minimal or, more generally, cmc and pmc submanifolds in Riemannian manifolds, is to use Simons type equations.

In 1968, J. Simons [**128**] discovered a fundamental formula for the Laplacian of the second fundamental form of a minimal submanifold in a Riemannian manifold and used it to characterize certain minimal submanifolds of a sphere and Euclidean space. One year later, K. Nomizu and B. Smyth [**112**] generalized Simons' equation in the case of cmc hypersurfaces in a space form and their result was then extended, in B. Smyth's work [**129**], to the more general case of pmc submanifolds in a space form. Over the years such formulas, nowadays called Simons type equations, were

used more and more often in studies on cmc and pmc submanifolds (see, for example, [**4, 7, 8, 16, 18, 28, 39, 123**]).

During the last three decades one could observe an ever growing interest in the study of certain fourth order partial differential equations, which generalize the notion of harmonic maps.

In their seminal paper [**49**], J. Eells and J. H. Sampson suggested the notion of *biharmonic maps* $\psi : M \to N$ between two Riemannian manifolds, defined as critical points of the *bienergy functional*

$$E_2(\psi) = \int_M |\tau(\psi)|^2 dv,$$

where $\tau(\psi) = \operatorname{trace} \nabla^{\psi} d\psi$ is the *tension field* of ψ, ∇^{ψ} being the connection in the pull-back bundle $\psi^{-1}TM$. If M is not a compact manifold, then biharmonic maps $\psi : M \to N$ are defined as solutions of the Euler-Lagrange equation $\tau_2(\psi) = 0$, where $\tau_2(\psi) = \Delta\tau(\psi) - \operatorname{trace} \bar{R}(d\psi, \tau(\psi))d\psi$ is the *bitension field* of ψ. It is easy to see that any harmonic map is biharmonic and that is why we are interested in *proper-biharmonic maps*, i.e., those biharmonic maps which are not harmonic.

A special case is that of *biharmonic Riemannian immersions*, or *biharmonic submanifolds*, i.e., those submanifolds for which the inclusion map is biharmonic. This definition of biharmonic submanifolds coincides, when working in Euclidean space (and only then), with that proposed by B.-Y. Chen [**31**], where a biharmonic submanifold is characterized by the fact that its mean curvature vector field is harmonic.

Although only non-existence results for proper-biharmonic submanifolds in Euclidean space were obtained (see, for example, [**34, 48, 91, 92, 106**]), when the ambient space is not flat, numerous examples and classification results for proper-biharmonic submanifolds were found in papers like [**12**]-[**14**], [**21**]-[**26**], [**44, 88, 100, 101, 104, 106**], [**117**]-[**119**], [**124**]-[**126**], and [**139**].

Our thesis is organized in two parts, the first one, that contains two chapters, being devoted to the study of pmc submanifolds, while in the second part, consisting of three chapters, we consider biharmonic submanifolds.

In the first chapter, *Reduction of codimension results and holomorphic differentials for surfaces with parallel mean curvature*, we present results from [**62**], [**77**], and [**78**] on pmc surfaces in complex, cosymplectic, and Sasakian space forms, respectively. In all these situations, we prove reduction of codimension theorems and also introduce holomorphic differentials that are then used to study the geometry of some of these surfaces.

The second chapter, *Simons type formulas and applications. Surfaces with parallel mean curvature and finite total curvature*, is devoted to the study of pmc submanifolds, this time of arbitrary dimension, in $M^n(c) \times \mathbb{R}$. We first prove two Simons type equations that are then used to obtain gap theorems for such submanifolds. We also consider pmc surfaces with finite total curvature and find a result on their compactness. The chapter ends with a classification result for helix pmc surfaces. These results were obtained in [**17**], [**63**], [**72**], [**74**], [**75**], and [**76**].

We begin dealing with biharmonic submanifolds in the third chapter, *Biharmonic and Biconservative Submanifolds in $M^n(c) \times \mathbb{R}$*, that is based on [**71**] and [**72**]. Here, we first present a gap theorem for pmc proper-biharmonic submanifolds in $M^n(c) \times \mathbb{R}$ and also classify pmc proper-biharmonic surfaces in this space. We then turn our attention to biconservative surfaces in $M^n(c) \times \mathbb{R}$, i.e., those surfaces for which the tangent part of the bitension field vanishes (we note that biconservative submanifolds have only very recently begun to be studied in articles like [**27, 80, 107, 108**]). We completely determine such surfaces that have parallel mean curvature vector field and obtain explicit examples of cmc biconservative surfaces, when $n = 3$. Also pmc biconservative surfaces with finite total curvature in Hadamard manifolds are considered and a compactness result is obtained, in the last part of the chapter.

In the fourth chapter, *Biharmonic submanifolds in Sasakian space forms*, we study biharmonic submanifolds in Sasakian space forms and obtain classification results, as well as explicit examples, for proper-biharmonic curves, proper-biharmonic Hopf cylinders over homogeneous real hypersurfaces in $\mathbb{C}P^n$, and 3-dimensional proper-biharmonic integral $\mathcal{C}$-parallel submanifolds in a 7-dimensional Sasakian space form. We also present a method to construct biharmonic anti-invariant submanifolds from biharmonic integral submanifolds. The results in this chapter first appeared in [**59**], [**60**], [**61**], and [**65**]-[**70**].

Biharmonic submanifolds in complex space forms are studied in the last chapter. First, are presented some general results on the biharmonicity of certain classes of submanifolds. We continue with a formula that relates the bitension fields of a submanifold in $\mathbb{C}P^n$ and its corresponding Hopf cylinder in $\mathbb{S}^{2n+1}$. Next, we prove a result on the biharmonicity of Clifford type submanifolds in $\mathbb{C}P^n$, while in the last part of the chapter we classify proper-biharmonic curves and pmc proper-biharmonic surfaces in $\mathbb{C}P^n$, and also 3-dimensional proper-biharmonic Lagrangian parallel submanifolds in $\mathbb{C}P^3$. This chapter contains results from [**64**], [**70**], and [**73**].

Whilst throughout the thesis we tried to offer the reader an image as complete as possible of our work, due to the need of keeping the presentation at a reasonable length, we were forced to skip many of the proofs and sometimes even not to mention some of our results that otherwise we consider interesting.

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