THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 19, Number 1/2018, pp. 45–52

NEW RHEOLOGICAL PROBLEMS INVOLVING GENERAL FRACTIONAL DERIVATIVES WITH NONSINGULAR POWER-LAW KERNELS

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Abstract. In this paper we propose the general Riemann-Liouville and Caputo-Liouville fractional derivatives with nonsingular power-law kernels, for the first time to our knowledge. New general laws of deformation within the framework of the general fractional derivatives are considered in detail. The creep and relaxation behaviors of the general fractional-order Voigt and Maxwell models are also obtained with the use of the Laplace transform. We provide the mathematical tools to describe the rheological phenomena of real materials with the memory effect.

Key words: general fractional derivatives with nonsingular power-law kernel, general fractional-order Voigt model, general fractional-order Maxwell model, Laplace transform, rheology.

1. INTRODUCTION

Fractional derivatives (FDs) with singular power-law kernels in the Riemann-Liouville and Caputo senses have been of increasing interest for scientists and engineers to develop the necessary mathematical models in diverse research areas such as control theory [1], mechanics [2], economy [3], finance [4], biology [5] and many others (see [6–13] and the cited references therein).

From the point of view of mathematics and physics, the new prospective of the applications of the FDs in linear viscoelasticity has been investigated in [14, 15]. For example, an idea for the fractional-order relaxation curve of the fluid was firstly considered in [16].

With the use of the Nutting's observation [17], the laws of deformation within the Riemann-Liouville [18], Liouville-Caputo [19–23], Caputo-Fabrizio [24], local [25] and others (see [26] and the cited references therein) were reported. The investigations of the hereditary elastic rheological model with the use of the Volterra integral equation were presented in [14, 15, 20–27]. The Maxwell and Voigt models within the different operators were proposed in [28–34].

Nowadays, there exists a new unsolved problem derived from the Nutting equation [17]. Motivated by the above-reported ideas, the main aim of this paper is to propose the general fractional derivatives (GFDs) with nonsingular power-law kernels to handle the general fractional-order Maxwell and Voigt models for the real materials with the nonsingular power-law phenomena.

This paper is organized as follows. In Section 2, we propose the Liouville-Caputo and Riemann-Liouville general fractional derivatives with non-singular power-law kernels. In Section 3, the rheological models involving general fractional derivatives with non-singular power-law kernels are considered. Finally, the conclusion is drawn in Section 4.

2. GFDS WITH NONSINGLAR POWER-LAW KERNELS

In this section, with the help of the concepts of the Liouville-Caputo and Riemann-Liouville FDs with the singular power-law kernel and their generalizations [5–11, 35–36], we start with the definitions of the Liouville-Caputo and Riemann-Liouville GFDs in a special function.

The Liouville-Caputo and Riemann-Liouville GFDs in a kernel function are defined as [35–36]:

$$\left(\mathbb{D}_{(\Xi)}^{C}\Theta\right)(\theta)=\int_0^{\theta}\Xi(\theta-t)\Theta^{(1)}(t)\,\mathrm{d}t \quad (\theta>0), \tag{1}$$

$$\left(\mathbb{D}_{(\Xi)}^{RL}\Theta\right)(\theta)=\frac{\mathrm{d}}{\mathrm{d}\theta}\int_a^{\theta}\Xi(\theta-t)\Theta(t)\,\mathrm{d}t \quad (\theta>0), \tag{2}$$

respectively, where $\mathrm{d}\Theta(\theta)/\mathrm{d}\theta=\Theta^{(1)}(\theta)$, $\Theta^{(1)}\in L_1^{loc}\left(\mathbb{R}_0^+\right)$, and $\Xi(\theta)$ is the kernel function.

When the nonsingular power-law kernel in Eq. (1) is given as:

$$\Xi(\theta)=\frac{\theta^{\nu}}{\Gamma(1+\nu)}, \tag{3}$$

the Riemann-Liouville type GFD of the function $\Theta(\theta)$ of order $(0<\nu<1)$ is defined by

$$\left({}_{0}^{RLT}\mathbb{D}_{\theta}^{(\nu)}\Theta\right)(\theta)=\frac{1}{\Gamma(1+\nu)}\frac{\mathrm{d}}{\mathrm{d}\theta}\int_0^{\theta}(\theta-t)^{\nu}\,\Theta(t)\,\mathrm{d}t \quad (\theta>0), \tag{4}$$

and the Liouville-Caputo type GFD of the function $\Theta(\theta)$ of order $(0<\nu<1)$ is defined by

$$\left({}_{0}^{CT}\mathbb{D}_{\theta}^{(\nu)}\Theta\right)(\theta)=\frac{1}{\Gamma(1+\nu)}\int_0^{\theta}(\theta-t)^{\nu}\,\Theta^{(1)}(t)\,\mathrm{d}t \quad (\theta>0). \tag{5}$$

The relationship between Eqs. (4) and (5) is given as:

$$\left({}_{0}^{RLT}\mathbb{D}_{\theta}^{(\nu)}\Theta\right)(\theta)=\left({}_{0}^{CT}\mathbb{D}_{\theta}^{(\nu)}\Theta\right)(\theta)+\frac{\theta^{\nu}\Theta(0)}{\Gamma(1+\nu)}. \tag{6}$$

Similarly, we define, corresponding to expressions of Eqs. (5) and (6), the following:

$$\left({}_{0}^{RLT}\mathbb{D}_{\tau}^{(\nu)}\Theta\right)(\tau)=\frac{1}{\Gamma(m+\nu)}\frac{\mathrm{d}^m}{\mathrm{d}\tau^m}\int_0^{\tau}(\tau-t)^{m+\nu-1}\,\Theta(t)\,\mathrm{d}t \quad (\tau>0), \tag{7}$$

$$\left({}_{0}^{CT}\mathbb{D}_{\tau}^{(\nu)}\Theta\right)(\tau)=\frac{1}{\Gamma(m+\nu)}\int_0^{\tau}(\tau-t)^{m+\nu-1}\,\Theta^{(m)}(t)\,\mathrm{d}t \quad (\tau>0), \tag{8}$$

where $m-1<\nu<m$.

The Laplace transforms of Eqs. (4) and (5) are as follows:

$$\mathfrak{A}\left[\left({}_{0}^{RLT}\mathbb{D}_{\tau}^{(\nu)}\Theta\right)(\tau)\right]=\frac{1}{s^{\nu}}\Theta(s), \tag{9}$$

$$\mathfrak{A}\left[\left({}_{0}^{CT}\mathbb{D}_{\tau}^{(\nu)}\Theta\right)(\tau)\right]=\frac{1}{s^{1+\nu}}\left(s\Theta(s)-\Theta(0)\right), \tag{10}$$

where the Laplace transform is defined by [27]:

$$\mathfrak{A}\left[\Phi(\tau)\right]=\Phi(s):=\int_0^{\infty}e^{-s\tau}\Phi(\tau)\,\mathrm{d}\tau. \tag{11}$$

The general fractional integral of the function $\Theta(\tau)$ of order $(0<\nu<1)$ is defined as:

$$\left({}_{a}^{RLT}\mathbb{I}_{\tau}^{(\nu)}\Theta\right)(\tau)=\frac{1}{\Gamma(-\nu)}\int_0^{\tau}\frac{\Theta(t)}{(\tau-t)^{1+\nu}}\,\mathrm{d}t \quad (\tau>0). \tag{12}$$

We call the new results as the generalized fractional calculus (GFC) with nonsingular power-law kernels.

3. RHEOLOGICAL MODELS INVOLVING GDFS WITH NONSINGULAR POWER-LAW KERNELS

In this section, we discuss the spring-dashpot elements, the creep and relaxation representations, the general fractional-order Voigt model (GFOVM), and the general fractional-order Maxwell model (GFOMM).

According to the Nutting equation in real materials [17], there is a power-law stress relaxation:

$$\sigma_\nu(\tau) = \mathrm{K}\tau^\nu \varepsilon_\nu(\tau), \tag{13}$$

where K is a model parameters, $\varepsilon_\nu(\tau)$ is the strain, σ_ν is the stress, and τ is the time.

With the use of Eqs. (4) and (5), the general fractional-order dashpot elements (GFODEs) follows the general fractional-order Newtonian law, whose creeping equations (CEs) are as follows:

$$\sigma_\nu(\tau) = \mathrm{K}\left({}^{RLT}_{\ 0}\mathbb{D}^{(\nu)}_{\tau}\varepsilon_\nu\right)(\tau), \tag{14}$$

$$\sigma_\nu(\tau) = \mathrm{K}\left({}^{CT}_{\ 0}\mathbb{D}^{(\nu)}_{\tau}\varepsilon_\nu\right)(\tau), \tag{15}$$

where K is the viscosity.

The spring element (SE) follows the Hooke's law, whose CE is given as:

$$\sigma_\nu(\tau) = \mathrm{H}\varepsilon_\nu(\tau), \tag{16}$$

where H is the Young's modulus of the material.

If $\sigma_\nu(0)$ and $\varepsilon_\alpha(0)$ represent the initial stress and strain conditions, respectively, then the creep compliance (CC) and relaxation modulus (RM) functions are given by:

$$J_\nu(\tau) = \varepsilon_\nu(\tau)/\sigma_\nu(0) \text{ and } G_\nu(\tau) = \sigma_\nu(\tau)/\varepsilon_\nu(0), \tag{17}$$

respectively.

Due to the Boltzmann superposition principle and causal histories for $\tau \in \mathbb{R}_0^+$ (see [14]), the creep and relaxation representations are as follows.

With the aid of Eq. (14), the CEs for creep and relaxation representations can be given though the hereditary integral equations of the Volterra type:

$$\varepsilon_\nu(\tau) = \sigma_\nu(0) J_\nu(\tau) + \int_0^\tau J_\nu(\tau - t)\left({}^{RLT}_{\ 0}\mathbb{D}^{(\nu)}_{t}\sigma_\nu\right)(t)\,\mathrm{d}t, \tag{18}$$

and

$$\sigma_\nu(\tau) = \varepsilon_\nu(0) G_\nu(\tau) + \int_0^\tau G_\nu(\tau - t)\left({}^{RLT}_{\ 0}\mathbb{D}^{(\nu)}_{t}\varepsilon_\nu\right)(t)\,\mathrm{d}t, \tag{19}$$

respectively.

Similarly, making use of Eq. (15), the CEs for the creep and relaxation representations can be applied by the consideration of the hereditary integral equations of the Volterra type:

$$\varepsilon_\nu(\tau) = \sigma_\nu(0) J_\nu(\tau) + \int_0^\tau J_\nu(\tau - t)\left({}^{CT}_{\ 0}\mathbb{D}^{(\nu)}_{t}\sigma_\nu\right)(t)\,\mathrm{d}t \tag{20}$$

and

$$\sigma_\nu(\tau) = \varepsilon_\nu(0) G_\nu(\tau) + \int_0^\tau G_\nu(\tau - t)\left({}^{CT}_{\ 0}\mathbb{D}^{(\nu)}_{t}\varepsilon_\nu\right)(t)\,\mathrm{d}t,, \tag{21}$$

respectively.

3.1. The GFOVMs with nonsingular power-law kernels

If the GFOVM consists of a HE and a GFODE in parallel [14, 25], then we have, by using Eqs. (14), (15), and (16), that

$$\sigma_\nu(\tau) = \mathrm{H}\varepsilon_\nu(\tau) + \mathrm{K}\left({}_{0}^{RLT}\mathbb{D}_{\tau}^{(\nu)}\varepsilon_\nu\right)(\tau), \tag{22}$$

$$\sigma_\alpha(\tau) = \mathrm{H}\varepsilon_\alpha(\tau) + \mathrm{K}\left({}_{0}^{CT}\mathbb{D}_{\tau}^{(\nu)}\varepsilon_\nu\right)(\tau), \tag{23}$$

respectively.

We now consider the creep behaviors of the GFOVMs as follows.

The system of the material is subjected to

$$\sigma_\nu(\tau) = \sigma_\nu(0)\vartheta(\tau), \tag{24}$$

where $\vartheta(\tau)$ represents the unit step function [14, 24] .

From Eqs. (22) and (24), the creeping differential equation (CDE) for the GFOVM is given by:

$$\sigma_\nu(0)\vartheta(\tau) = \mathrm{H}\varepsilon_\nu(\tau) + \mathrm{K}\left({}_{0}^{RLT}\mathbb{D}_{\tau}^{(\nu)}\varepsilon_\nu\right)(\tau), \tag{25}$$

which leads to the CC function:

$$J_\nu(\tau) = \frac{1}{\mathrm{H}} E_\nu\left(-\frac{\mathrm{K}}{\mathrm{H}}\tau^\nu\right), \tag{26}$$

where the Mittag-Leffler function is defined as [6–8, 14]:

$$E_\nu(\vartheta) = \sum_{\eta=0}^{\infty} \frac{\vartheta^{\eta\nu}}{\Gamma(\eta\nu+1)}. \tag{27}$$

In a similar way, from Eqs. (23) and (24), the CDE for the GFOVM is presented as:

$$\sigma_\nu(0)\vartheta(\tau) = \mathrm{H}\varepsilon_\nu(\tau) + \mathrm{K}\left({}_{0}^{CT}\mathbb{D}_{\tau}^{(\nu)}\varepsilon_\nu\right)(\tau), \tag{28}$$

which yields the CC function:

$$J_\nu(\tau) = \frac{1}{\mathrm{H}} E_\nu\left(-\frac{\mathrm{K}}{\mathrm{H}}\tau^\nu\right) + \frac{\mathrm{K}}{\mathrm{H}^2} E_{\nu,\nu+1}\left(-\frac{\mathrm{K}}{\mathrm{H}}\tau^\nu\right), \tag{29}$$

where the two-parameter Mittag-Leffler function is defined as [6, 14]:

$$E_{\alpha,\upsilon}(\vartheta^\nu) = \sum_{\eta=0}^{\infty} \frac{h^{\eta\nu}}{\Gamma(\eta\nu+\upsilon)}. \tag{30}$$

Figure 1 illustrates the creep responses of the GFOVMs via the nonsingular power-law kernel for the parameter $\nu = 0.8$.

We present the relaxation behaviors of the GFOVMs as follows.

The system of the material is subjected to

$$\varepsilon_\nu(\tau) = \varepsilon_\nu(0)\vartheta(\tau). \tag{31}$$

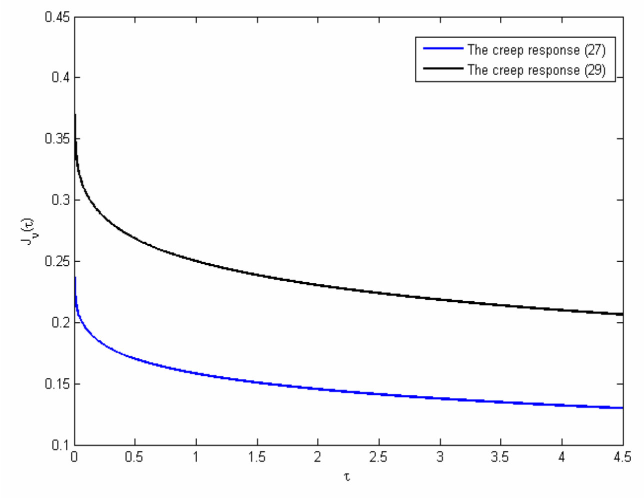

Fig. 1 – The creep responses of the GFOVMs for the parameter $\nu = 0.8$.

From Eqs. (22) and (31), the relaxation differential equation (RDE) for the GFOVM is given as:

$$\sigma_\nu(\tau) = \mathrm{H}\varepsilon_\nu(0)\vartheta(\tau) + \frac{\mathrm{K}\varepsilon_\nu(0)\tau^\nu}{\Gamma(1+\nu)}, \tag{32}$$

which leads to the RM function:

$$G_\nu(\tau) = \mathrm{H} + \frac{\mathrm{K}\tau^\nu}{\Gamma(1+\nu)}. \tag{33}$$

In a similar manner, due to Eqs. (23) and (31), the RDE for the GFOVM can be read as:

$$\sigma_\nu(\tau) = \mathrm{H}\varepsilon_\nu(0), \tag{34}$$

which yields the RM function:

$$G_\nu(\tau) = \mathrm{H}. \tag{35}$$

Figure 2 shows the relaxation responses of the GFOVMs via the nonsingular power-law kernel for the parameter $\nu = 0.8$.

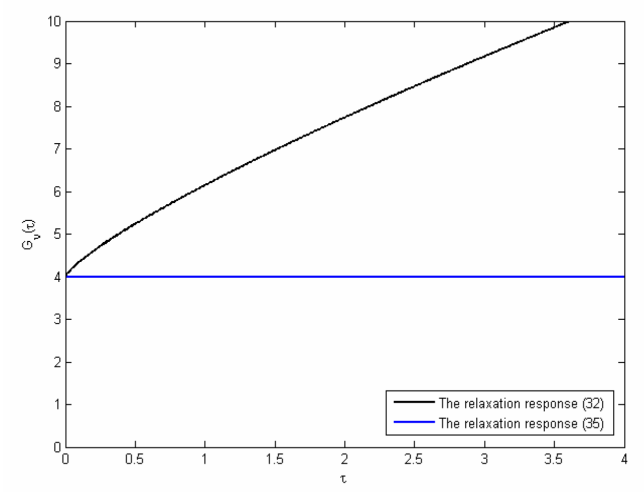

Fig. 2 – The relaxation responses of the GFOVMs for the parameter $\nu = 0.8$.

3.2. The GFOMMs with nonsingular power-law kernels

If the GFOMM consists of a HE and a GFODE in series [14, 25], then we have, by using Eqs.(14), (15), and (16), that

$$\left({}_{0}^{RLT}\mathbb{D}_{\tau}^{(\nu)}\varepsilon_{\nu}\right)(\tau)=\frac{\sigma_{\nu}(\tau)}{\mathrm{K}}+\frac{1}{\mathrm{H}}\left({}_{0}^{RLT}\mathbb{D}_{\tau}^{(\nu)}\sigma_{\nu}\right)(\tau), \tag{36}$$

$$\left({}_{0}^{CT}\mathbb{D}_{\tau}^{(\nu)}\varepsilon_{\nu}\right)(\tau)=\frac{\sigma_{\nu}(\tau)}{\mathrm{K}}+\frac{1}{\mathrm{H}}\left({}_{0}^{CT}\mathbb{D}_{\tau}^{(\nu)}\sigma_{\nu}\right)(\tau), \tag{37}$$

respectively.

We consider the creep behaviors of the GFOMMs as follows.

With the aid of Eqs. (24) and (36), the CDE for the GFOMM is given by

$$\left({}_{0}^{RLT}\mathbb{D}_{\tau}^{(\nu)}\varepsilon_{\nu}\right)(\tau)=\frac{\sigma_{\nu}(0)\vartheta(\tau)}{\mathrm{K}}+\frac{1}{\mathrm{H}}\frac{\sigma_{\nu}(0)\tau^{\nu}}{\Gamma(1+\nu)}, \tag{38}$$

which leads to the CC function:

$$J_{\nu}(\tau)=\frac{1}{\mathrm{H}}+\frac{1}{\mathrm{K}}\frac{\tau^{-(\nu+1)}}{\Gamma(-\nu)}. \tag{39}$$

From Eqs. (24) and (37), the CDE for the GFOMM is represented as:

$$\left({}_{0}^{CT}\mathbb{D}_{\tau}^{(\nu)}\varepsilon_{\nu}\right)(\tau)=\frac{\sigma_{\nu}(0)\vartheta(\tau)}{\mathrm{K}}, \tag{40}$$

which reduces to the CC function:

$$J_{\nu}(\tau)=\frac{1}{\mathrm{H}}+\frac{1}{\mathrm{K}}\frac{\tau^{-(\nu+1)}}{\Gamma(-\nu)}. \tag{41}$$

Figure 3 displays the creep responses of the GFOMMs via the nonsingular power-law kernel for the parameter $\nu=0.8$.

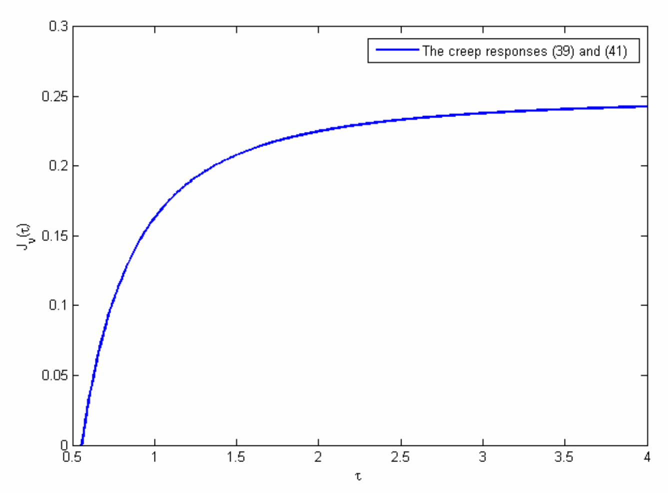

Fig. 3 – The creep responses of the GFOMMs for the parameter $\nu=0.8$.

Let us consider the relaxation behaviors of the GFOMMs as follows. In view of Eqs. (39) and (44), the RDE for the GFOMM can be read as:

$$\frac{\varepsilon_{\nu}(0)\vartheta(\tau)\tau^{\nu}}{\Gamma(1+\nu)}=\frac{\sigma_{\nu}(\tau)}{\mathrm{K}}+\frac{1}{\mathrm{H}}\left({}_{0}^{RLT}\mathbb{D}_{\tau}^{(\nu)}\sigma_{\nu}\right)(\tau), \tag{42}$$

which leads to the RM function:

$$G_\nu(\tau) = \mathrm{K} E_{\nu-1,\nu}\left(-\frac{\mathrm{K}}{\mathrm{H}}\tau^{\nu-1}\right). \tag{43}$$

From Eqs. (39) and (45), the RDE is as follows:

$$\frac{\sigma_\nu(\tau)}{\mathrm{K}} + \frac{1}{\mathrm{H}}\left({}_0^{CT}\mathbb{D}_\tau^{(\nu)}\sigma_\nu\right)(\tau) = 0, \tag{44}$$

which leads to the RM function:

$$G_\nu(\tau) = \mathrm{K} E_{\nu,\nu-1}\left(-\frac{\mathrm{K}}{\mathrm{H}}\tau^{\nu}\right). \tag{45}$$

Figure 4 demonstrates the relaxation responses of the GFOMMs via the nonsingular power-law kernel for the parameter $\nu = 0.8$.

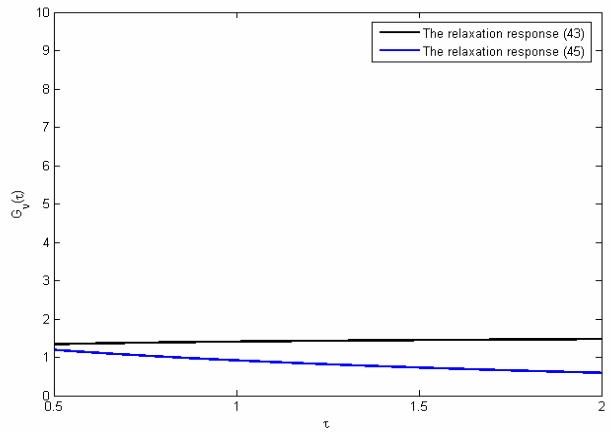

Fig. 4 – The relaxation responses of the GFOMMs for the parameter $\nu = 0.8$.

4. CONCLUSION

In the present work, we proposed the nonlocal Liouville-Caputo and Riemann-Liouville GFDs with nonsingular power-law kernels to model the rheological phenomena in real materials with memory effects, for the first time to our knowledge. The solutions for the creep and relaxation differential equations of the GFOVMs and GFOMMs were also discussed with the aid of the Laplace transform. The GFC may open a new prospective for handling various nonsingular power-law phenomena in science and engineering.

ACKNOWLEDGEMENTS

This work is supported by the State Key Research Development Program of the People's Republic of China (Grant No. 2016YFC0600705), the Natural Science Foundation of China (Grant No.51323004), and the Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD2014).

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Received June 13, 2017