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DYNAMICAL ANALYSIS OF AN NON-ŠIL'NIKOV CHAOTIC SYSTEM

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A system with non-existence of Šil'nikov sense chaos was constructed in this paper. To further understand the complex dynamics of the system, some basic behaviors such as the largest Lyapunov exponent, bifurcation diagram, Poincaré mapping were revealed by rigorous numerical simulations. Interestingly, the chaotic attractors can coexist with quasi-periodicity attractors in presented system, which has rarely been reported in three-dimensional autonomous systems. Next, a three-dimensional (3D) sphered ultimate bound and positive invariant set were derived for the positive values of its parameters a, b Then, the stability of non-hyperbolic equilibrium is investigated. Finally, the existence of singularly degenerate heteroclinic cycles for a suitable choice of the parameters is studied.

Key words: Šil'nikov sense chao, degenerate heteroclinic cycles, coexisting attractor.

1. INTRODUCTION

Since the now-chaotic attractor was presented by Lorenz in 1963 [1], chaos phenomenon has received more and more attention from scientific community, meanwhile, many more chaotic systems were developed after Lorenz system, such as Rössler system [2], Chua system [3], Chen system [4], Lü system [5], Liu system [6], the Lorenz system family [7], the conjugate Lorenz-type system [8], and so on. The above chaotic systems are all belong to hyperbolic type of chaotic systems. The hyperbolic type of chaotic systems are called Šil'nikov sense chaos, which exists heteroclinic orbit or homoclinic orbit. According to one of the commonly agreed-upon analytic criteria for proving chaos in autonomous systems in [9], which requires the systems to have at least one unstable saddle-focus equilibrium point. So, the chaotic system without saddle-focus type of equilibrium is called non-existence of Šil'nikov sense chaos. Although most chaotic systems belong to Šil'nikov chaos, there are still many non-Šil'nikov chaos. For instance, a class of systems without equilibrium points were proposed in [10–14], Yang found a chaotic system with one saddle and two stable node-foci [15], Yang also introduced a chaotic system with two stable node-foci [16], Wang reported a chaotic system with only one stable node-focus equilibrium [17], and a catalog of chaotic systems with a line equilibrium were presented by Jafari in [18]. Later, literatures [19, 20] introduced a new class of chaotic systems with a circular equilibrium. Recently, Li constructed a simple chaotic system with non-hyperbolic equilibria [21]. More recently, Chen and Han reported an interesting three-dimensional quadratic chaotic system with two equilibrium points of saddle-focus type, but it does not belong to Šil'nikov type chaos in view of the algebraic condition given by Elhadj [22]. It is very important to note that Wei and Yang introduced the generalized Sprott C system with only two stable equilibria, which reveals the influence of initial condition on the dynamics of system with the fixed parameters values [23]. Therefore, it is interesting to ask whether or not there is another 3D system with non-existence of Šil'nikov chaos, whose dynamical behavior is closely related to initial value. This problem will be investigated in the present paper. A thorough study of such kind of chaotic systems may be beneficial to understand the complicated mechanisms of chaos. For such kind of chaotic systems, there seems to be no study on chaotic encryption or decryption, so it may be more security.

Motivated by the above works, we introduce non-existence of Šil'nikov existence sense chaos with different types of equilibria and initial values. The system can generate a double-scroll chaotic attractor, which can coexist with quasi-periodicity attractor. First of all, we use Lyapunov stability theory to estimate the precise bound of this chaotic system. In addition, non-chaotic behavior in the system was discussed.

Moreover, the stability of nonhyperbolic equilibrium is studied. At last, the existence of singularly degenerate heteroclinic cycles for a suitable choice of the parameters is revealed.

2. AN NON-ŠIL'NIKOV CHAOTIC SYSTEM

The proposed chaotic system is expressed as

$$\begin{aligned} \dot{x} &= a(-x-y), \\ \dot{y} &= -xz, \\ \dot{z} &= c+xy-bz, \end{aligned} \tag{1}$$

where dot denotes derivative with respect to time t , a, b, c are real numbers. It is easy to see that system (1) is globally uniformly and asymptotical stable about its equilibrium point $S_0 = (0,0,\frac{c}{b})$ if $a > 0$ and $bc < 0$. Actually, this is can be demonstrated by constructing Lyapunov function in the following form

$$V_1(x,y,z) = \frac{1}{2}x^2 - \frac{ab}{2c}[y^2 + (z - \frac{c}{b})^2], \tag{2}$$

which leads to

$$\dot{V}_1(x,y,z) = -a[x^2 - \frac{b^2}{c}(z - \frac{c}{b})^2]. \tag{3}$$

and by setting

$$\left\{(x,y,z)\middle|\dot{V}_1(x,y,z) = 0\right\} = \left\{(x,y,z)\middle|x = 0, z = \frac{c}{b}, y \in R\right\}.$$

Which does not contain a nontrivial trajectory of system (1). We can derive from the Krasnoselskii theorem that system (1) is globally uniformly and asymptotically stable about the equilibrium S_0. This means that the system (1) is not chaotic in the parameter region $\{(a,b,c)\,|\,a > 0, b > 0, c < 0\}$.

2.1. Coexisting attractors with different types of equilibria and different initial values

If $c > 0$, system (1) has three equilibria $S_0 = (0,0,\frac{c}{b})$, $S_+ = (\sqrt{c},-\sqrt{c},0)$ and $S_- = (-\sqrt{c},\sqrt{c},0)$. When $c \le 0$, $S_0 = (0,0,\frac{c}{b})$ is unique equilibrium. Although the system has the simple form, it can display complicated and unusual dynamical behaviors. Next, we investigate complex dynamics of system (1) in different initial values.

(a) When parameter $a = 2, b = 1, c = 5.5$, it is clearly that system (1) exists three equilibria

$$S_0(0,0,5.5)\ S_{\pm}(\pm\sqrt{5.5},\mp\sqrt{5.5},0)$$

And the corresponding eigenvalues are $\lambda_1(S_0) = -4.4641, \lambda_2(S_0) = 2.4641, \lambda_3(S_0) = -1$ and

$$\lambda_1(S_\pm) = -2.9694, \lambda_2(S_\pm) = -0.0153 + 2.7219i, \lambda_3(S_\pm) = -0.0153 - 2.7219i.$$

Obviously, system (1) has one saddle and two stable node-foci equilibra, and system (1) is not under the class of Šil'nikov sense chaos.

(1) For initial values $(0.1, 0, 9)$, the attractor is quasi-periodicity. The Lyapunov exponents of system (1) are $L_1 = 0$, $L_2 = 0$, $L_3 = -2.9696$.

(2) System (1) displays chaotic behavior with initial values $(0.1, 0, 0.1)$. The Lyapunov exponents of system (1) are found to be $L_1 = 0.2144, L_2 = 0, L_3 = -3.2132$ .Thus, system (1) can display a chaotic attractor.

(b) With $a = 2$, $b = 1$, $c = 6$, it is easy to see the system (1) has three equilibria $S_0(0,0,6)$

$$S_{\pm}(\pm\sqrt{6},\ \mp\sqrt{6},0).$$

Whose corresponding characteristic value are $\lambda_1(S_0)=-4.4641,\ \lambda_2(S_0)=2.4641,\ \lambda_3(S_0)=-1$ and

$$\lambda_1(S_{\pm})=-3,\ \lambda_2(S_{\pm})=2.8284\mathrm{i},\ \lambda_3(S_{\pm})=-2.8284\mathrm{i}\ .$$

(1) Under initial conditions $(20, 0.3, 0)$ the Lyapunov exponents of system (1) $L_1=0, L_2=0, L_3=-2.999.$ Thus, system (1) is quasi-periodicity.

(2) For the initial values $(19, 0.3, 0)$, the Lyapunov exponents of system (2.1) are found to be $L_1=0.05, L_2=0, L_3=-3.0517.$ Therefore, system (1) indeed exists chaotic attractor with saddle and two non-hyperbolic equilibra. The corresponding view of attractor is displayed in Fig. 1 and Fig. 2.

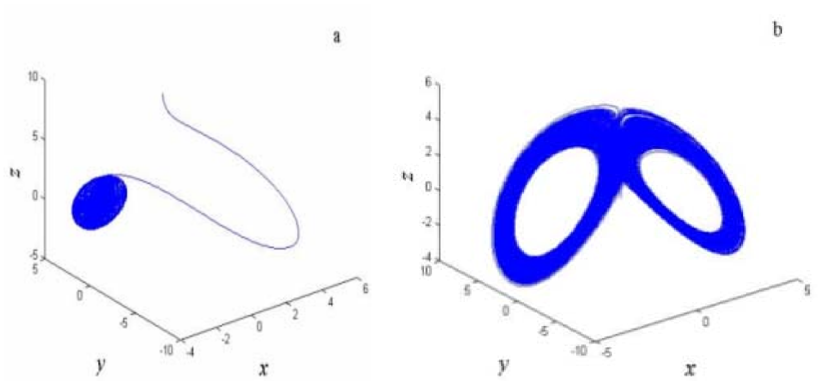

Fig. 1– a) Quasi-periodicity attractor with initial values (0.1, 0, 9);
b) chaotic attractor with initial values (0.1, 0, 0.1) for a = 2, b = 1, c = 5.5.

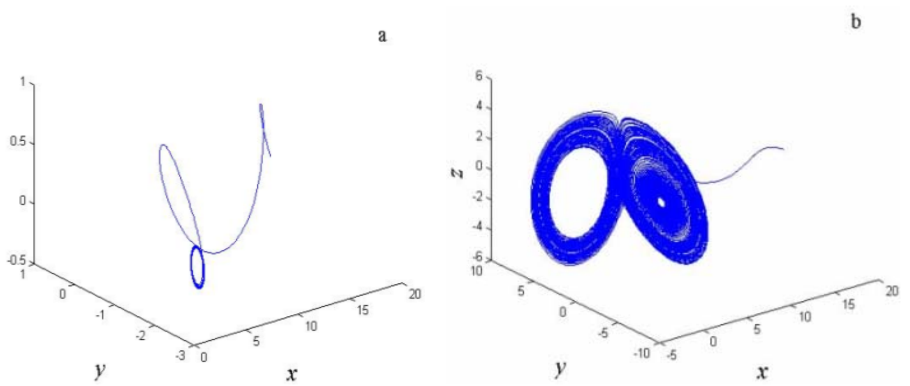

Fig. 2– a) For a = 2, b = 1, c = 6 a) initial values (20, 0.3, 0), the quasi-periodicity attractor of system (1); b) initial values (19, 0.3, 0), chaotic attractor of system (1).

Remark 1. From Fig. 1 and Fig. 2, it can be found that when we fix system parameters and change initial values, the dynamics properties of the system make large variations. Of particular interest is the fact that chaos coexists with quasi-periodicity attractors.

Remark 2. Equilibrium point plays an important role in their properties, however, the whole structure of chaotic attractors is completely determined by it. Šil'nikov criteriais is sufficient but certainly not necessary for emergence of chaos.

2.2. Poincaré mapping, bifurcation diagram, Lyapunov exponent spectrum

The numerical features of the new chaotic attractor can be further illustrated by the Poincaré mapping, are shown in Fig. 3 and Fig. 4, respectively.

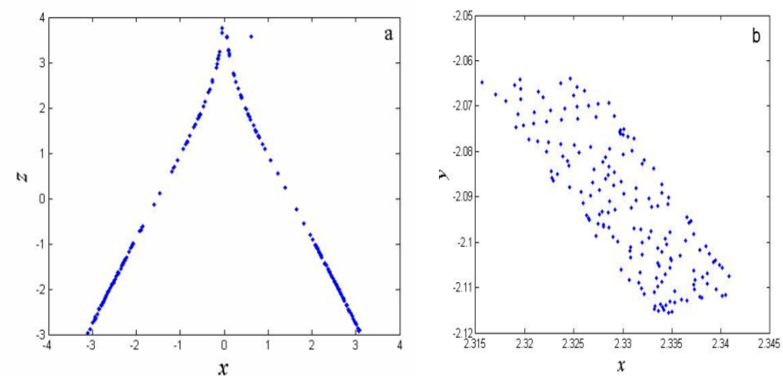

Fig. 3 – a) For $a = 2$, $b = 1$, $c = 5.5$, initial values (0.1, 0, 0.1) projection on *x-z* Poincaré map;
b) under $a = 2$, $b = 1$, $c = 6$, initial values (19, 0.3, 0), projection on *x-y* Poincaré map .

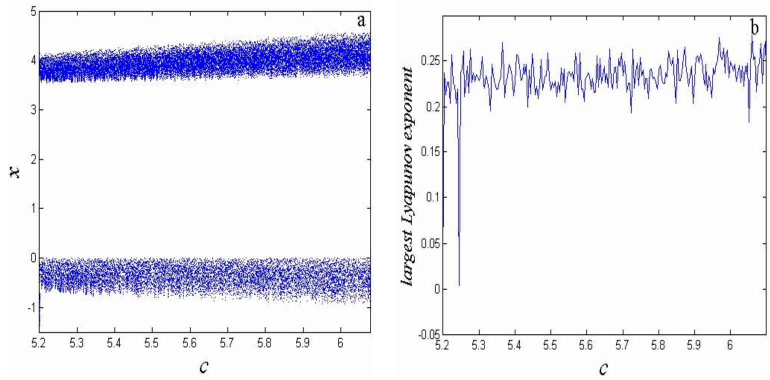

Fig. 4– Parameters $a = 2$, $b = 1$, and $c \in (5.2, 6.1)$ under initial condition (0.1, 0, 0.1):
a) bifurcation diagram; b) the largest Lyapunov exponent versus the parameter c.

2.3. Symmetry, invariance and dissipativity

It is easy to see that system (1) is invariant under the coordinates transformation $(x, y, z) \to (-x, -y, z)$, that is, the system has rotation symmetry around the $z-$axis.

The divergence of flow of the dynamic system (1) is

$$\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -a - b \tag{4}$$

Hence, system (1) is dissipative for $a + b > 0$. That is to say, a volume element $V(t) = V_0 \exp((-a-b)t)$ becomes smaller in time t. This means that each volume containing the trajectories of the dynamical system shrinks to zero as $t \to \infty$ at an exponential rate. As a result, system orbits finally are restricted to a subset whose volume is zero and their asymptotical behavior and fixed on an attractor.

2.4. The boundedness of the solutions of the system (1)

For the boundedness of system (1), we derive the following result.

THEOREM 1. *Suppose that* $a > 0$ *and* $b > 0$. *Then, all orbits of system (1) are trapped in bounded region including chaotic attractors.*

Proof. For any one solution x, y, z of system (1), define the following Lyapunov function

$$V_2(x, y, z) = \frac{1}{2}[x^2 + y^2 + (z+a)^2] \tag{5}$$

Computing the derivative of V with respect to time t along the solution of system (1), we obtain

$$\dot{V}_2(x,y,z) = -ax^2 + (c-ab)z - bz^2 + ac$$
$$= -ax^2 - b(z - \frac{c-ab}{2b})^2 + \frac{(c+ab)^2}{4b}. \quad (6)$$

Let $e_0 > 0$ be so sufficiently large that for all (x,y,z) satisfying $V(x,y,z) = e$ with $e > e_0$ one has

$$ax^2 + b(z - \frac{c-ab}{2b})^2 > \frac{(c+ab)^2}{4b}. \quad (7)$$

Hence, on the surface $\{(x,y,z) | V_2(x,y,z) = e\}$ with $e > e_0$, one has $\dot{V}_2(x,y,z) < 0$,which indicates that the $\{(x,y,z) | V_2(x,y,z) \le e\}$ is a trapped region of all solutions of system (1). This completes the proof. For the numerical simulation of theorem 1, as shown in Fig. 5.

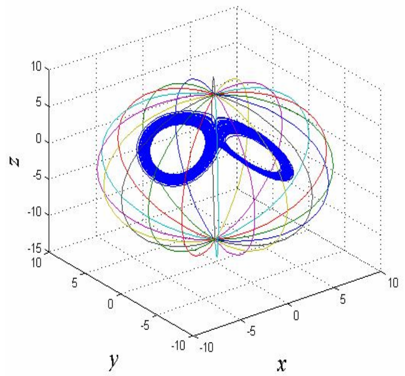

Fig. 5 – The orbits or system (1) for parameters $a = 2$, $b = 1$, $c = 5.5$, which obviously lie in a bounded sphere.

2.5. Non-chaotic behavior in the system (1)

It is straightforward to show that the whole structure of chaotic attractors is not completely determined by the knowledge of fixed points and their properties. We show here there are some nonchaotic parameter regions. This implies that the system (1) cannot have chaotic solutions in some cases. Next, we prove the following Theorem 2.

THEOREM 2. *If one of the following conditions is satisfied*

(1) $a = 0$

(2) $a < 0,\ c < 0,\ a + b > 0$

(3) $a > 0,\ \ c > 0,\ a + b < 0$.

Then the system (1) *is not chaotic.*

Proof. It is obviously that system (1) is not chaotic under assumption (1). Then, we only consider the case $a \ne 0$. According to the first equation of (1), we can get the

$$\ddot{x} = -a\dot{x} - a\dot{y} = -a\dot{x} + axz \quad (8)$$

and

$$axz = a\dot{x} + \ddot{x} \quad (9)$$

Calculating the derivative of equation (8), we have

$$\dddot{x} = -a\ddot{x} + a\dot{x}z + ax\dot{z} \quad (10)$$

Substituting $\dot{z} = c + xy - bz$ into (10), one gets

$$\dddot{x} = -a\ddot{x} + a\dot{x}z + acx + ax^2y - abxz \quad (11)$$

Multiply both sides of (11) by x, we get the equation

$$x\ddot{x} = -ax\ddot{x} + a\dot{x}xz + acx^2 + ax^3y - abx^2z\ . \tag{12}$$

From $\dot{x} = a(-x-y)$, one derives

$$y = \frac{-ax - \dot{x}}{a}\ . \tag{13}$$

Substituting expression (9) and (13) into expression (12), one obtains

$$(a+1)x\ddot{x} - \dot{x}\ddot{x} + x^3\dot{x} + bx\dot{x} + abx\dot{x} = a\dot{x}^2 + acx^2 - ax^4\ . \tag{14}$$

Integrating equation (14), we have

$$(a+1)x\ddot{x} - \frac{a+2}{2}\dot{x}^2 + \frac{1}{4}x^4 + bx\dot{x} + \frac{1}{2}abx^2 = \int_0^t[(a+b)\dot{x}^2 + acx^2 - ax^4]\mathrm{d}t + C\ . \tag{15}$$

Here C is a constant and $t > 0$. The left side of polynomial equations (15) can be simplified to

$$(a^3x + a^2x - a - b)(x+y) + (a+1)x^2z - \frac{a^2(a+1)}{2}(x+y)^2 + \frac{1}{4}x^4 + \frac{1}{2}abx^2. \tag{16}$$

When conditions (2) or (3) of theorem 2 are satisfied, the (16) is monotone as a function of time, and has a limit $L \in R$ as $t \to \infty$ if L is finite, then any attractor for the equation lies on the surface (16) and is not chaotic by virtue of the Poincareé–Bendixson theorem. If $L = \pm\infty$, then at least one of the three variables is not bounded and system (1) is not chaotic.

3. THE STABILITY OF NONHYPERBOLIC EQUILIBRIUM

According to the Routh-Hurwitz stability criterion, the stability of hyperbolic equilibrium is easily obtained. So we only consider the stability of nonhyperbolic equilibrium. If $c = 0$, the system has a unique equilibrium $S_0 = (0, 0, 0)$. The Jacobian matrix of the system (1) at the origin S_0

$$\mathbf{A} = \begin{pmatrix} -a & -a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -b \end{pmatrix},$$

with the characteristic equation

$$\lambda(\lambda + b)(\lambda + a) = 0. \tag{17}$$

Obviously, Eq. (17) has three roots $\lambda_1 = 0, \lambda_2 = -a$ and $\lambda_3 = -b$, with the corresponding eigenvectors being $(-1,1,0)^T$, $(1,0,0)^T$, $(0,0,1)^T$, respectively.

THEOREM 3. *Assume that* $a > 0, b > 0$ *and* $c = 0$, *then nonhyperbolic equilibrium* S_0 *is asymptotically stable.*

Proof. From the above discussion, one sees that the equilibrium S_0 is nonhyperbolic with three eigenvalues $0, -a$ and $-b$. Next, we will investigate the stability of S_0 by using the center manifold theorem [24].

Let us introduce the transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

Transform system (1) into the following form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{z}_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -b \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} (x_1 - y_1)z_1 \\ (x_1 - y_1)z_1 \\ x_1(y_1 - x_1) \end{pmatrix} \tag{18}$$

According to the center manifold theorem, the stability of S_0 can be determined by studying fist-order ordinary differential equations on a center manifold, which can be represented as a graph over the variables x_1 as bellow

$$W^c(0)=\left\{\begin{array}{l}(x_1,y_1,z_1)\in \mathrm{R}^3 \mid y_1=h_1(x_1), z_1=h_2(x_1), |x_1|<\delta, h_1(0)=0, h_2(0)=0,\\ h_1'(0)=0, h_2'(0)=0\end{array}\right\}$$

with δ sufficiently small.

Assume that

$$y_1=h_1(x_1)=a_1x_1^2+b_1x_1^3+O(x_1^4),\ z_1=h_2(x_1)=a_2x_1^2+b_2x_1^3+O(x_1^4). \quad (19)$$

Thus, the center manifold must satisfy

$$\begin{aligned} &h_1'(x_1)(x_1-h_1(x_1)h_2(x_1))+ah_1(x_1)-(x_1-h_1(x_1)h_2(x_1))=0,\\ &h_2'(x_1)(x_1-h_1(x_1)h_2(x_1))+bh_2(x_1)-x_1(h_1(x_1)-x_1)=0.\end{aligned} \quad (20)$$

Substituting expression (19) into Eq.(20) and equating the coefficients of x_1^2 and x_1^3 on both side, one obtains that

$$-ab_1+a_2=0,\ -aa_1=0,\ ba_2+1=0,\ a_1-bb_2=0. \quad (21)$$

By Solving equations (21), we find

$$a_1=0,\ b_2=0,\ a_2=-\frac{1}{b},\ b_1=-\frac{1}{ab} \quad (22)$$

and hence

$$h_1(x_1)=-\frac{1}{ab}x_1^2+O(x_1^4),\ h_2(x_1)=-\frac{1}{b}x_1^2+O(x_1^4). \quad (23)$$

Substituting expression (23) into expressions (19) and (18), one has the vector field reduced to the center manifold

$$\dot{x}_1=-\frac{1}{b}x_1^3-\frac{1}{ab^2}x_1^4+O(x_1^5). \quad (24)$$

Therefore, one can deduce that when a, $b>0$ the equilibrium S_0 is asymptotically stable. For the conclusions of theorem 3, refer to the simulation result in Fig. 6.

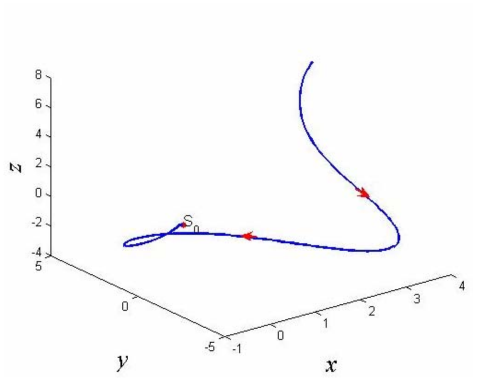

Fig. 6 – The phase portrait of system (1) with $a = 2$, $b = 1$, $c = 0$.

4. SINGULARLY DEGENERATE HETEROCLINIC CYCLES

For $b = c = 0$ the system (1) becomes the following form

$$\begin{aligned}\dot{x}&=a(-x-y),\\ \dot{y}&=-xz,\\ \dot{z}&=xy,\end{aligned} \quad (25)$$

which has the line of equilibria $(0,0,z)$, $z \in R$. Notice one is considering $a > 0$. By linearizing system (25) at the equilibrium point (0, 0, z) one obtains the Jacobian matrix

$$J = \begin{bmatrix} -a & -a & 0 \\ -z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

whose characteristic equation is

$$\lambda^3 + a\lambda^2 - az\lambda = 0. \quad (26)$$

Therefore, the eigenvalues

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 + 4az}}{2}, \quad \lambda_3 = 0. \quad (27)$$

With the corresponding eigenvectors given as follows

$$v_1 = (\frac{a - \sqrt{a^2 + 4az}}{2z}, 1, 0), \qquad v_2 = (\frac{a + \sqrt{a^2 + 4az}}{2z}, 1, 0), \qquad v_3 = (0, 0, 1). \quad (28)$$

Providing $z < -\frac{a}{4}$, the eigenvalues $\lambda_{1,2}$ are complex with the negative real part. Considering also the corresponding eigenvectors (28), this means that the solutions locally spiraling toward the equilibrium point $Q = (0,0,z)$ on a surface tangent to the plane spanned by the eigenvectors $v_{1,2}$, hence in direction normal to the z-axis. When $-\frac{a}{4} < z < 0$, the eigenvalues $\lambda_{1,2}$ are real and negative. Therefore, trajectories move toward to z-axis without spiraling. If $z > 0$, the eigenvalues $\lambda_{1,2}$ are real with opposite signs. Then taking into account the eigenvalues $v_{1,2}$, the system has a normally hyperbolic saddle at the point $P = (0,0,z)$. In the specific case in which $z = 0$ the equilibrium point $(0,0,0)$ is more degenerated, having two vanishing eigenvalues.

By above analysis and carefully numerical study (see Fig. 7) of the solutions of system (1) with $b = c = 0$ and $a > 0$ has been performed, which clearly reveals that the system proposes an infinite set of singularly degenerate heteroclinic cycles. Each one of these cycles is formed by one of the one-dimensional unstable manifolds of the saddle P, which connects P with normally hyperbolic focus Q, as $t \to +\infty$. As the system presents an infinite number of normally hyperbolic saddles P and foci Q, there exists an infinite set of singularly degenerate heteroclinic cycles. In Fig. 7a, some of them are shown: for each initial condition considered sufficiently close to the saddle P at the z-axis, a singularly degenerate heteroclinic cycle is created. According to Fig. 7b, we also observe that the saddles P and the stable focus Q extend to infinity on the negative and positive parts of z-axis.

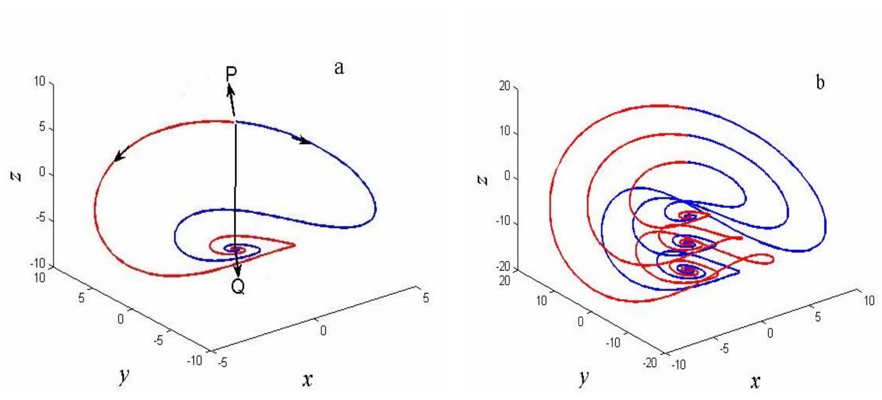

Fig. 7 – Singularly degenerate heteroclinic cycles of system (1) with (a, b, c) = (2, 0, 0):
a) initial value (0.07, –0.001, 7.001) and (0.07, –0.001, 7.001); b) initial values (0.02, 0.02, z(0)) and (–0.02, –0.02, z(0)), where z(0) ∈ {6.001, 12.001, 18.001}.

5. CONLUSIONS

An simple system with one saddle and two stable node-foci equilibria or with one saddle and two non-hyperbolic equilibria that coexists chaos and quasi-periodic torus has indroduced in this paper. A significant dynamics of this system is closely related to initial conditions. Abundant and complex dynamical behavors has been completely and thoroughly investigated. We hope that the finding discussed in this paper can provide some light for further exploiting the dynamics of non-Šil'nikov chaotic systems.

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